An Introduction to Quantum Field Theory (Peskin and Schroeder)

参考本：

- [1] Feynman ルールの証明はせずに摂動計算の練習
- [2] 非可換ゲージ理論について分かりやすく書いてある
- [3] 標準模型の Feynman rule 一覧．Higgs field ϕ の定義が異なるので注意が必要

Bibliography

[1] David Griffiths. *Introduction to elementary particles*. John Wiley & Sons, 2020.

[2] 坂本眞人. **量子力学選書 場の量子論-不変性と自由場を中心にして**. Shokabo Publishing Co., Ltd., 2020.

[3] Jorge C Romao and Joao P Silva. "A resource for signs and Feynman diagrams of the Standard Model". In: *International Journal of Modern Physics A* 27.26 (2012), p. 1230025.

[4] Abhijit Sen and Zurab K Silagadze. "Two-photon decay of P-wave positronium: a tutorial". In: *Canadian Journal of Physics* 97.7 (2019), pp. 693–700.

[5] David J Gross and Frank Wilczek. "Asymptotically free gauge theories. II". In: *Physical Review D* 9.4 (1974), p. 980.

目次

Part I

Feynman Diagrams and Quantum Electrodynamics

Fourier 変換

場の Fourier 変換は xxi のように

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\phi(k), \quad \phi(k) = \int d^4x\, e^{ik\cdot x}\phi(x)$$

と定める．Fermion の場合は

$$\psi(p) = \int d^4x\, e^{ip\cdot x}\psi(x), \quad \bar{\psi}(p) = \int d^4x\, e^{-ip\cdot x}\bar{\psi}(x)$$

とする[*1]．Propagator は

$$\begin{aligned}
\overbrace{\psi(p)\bar{\psi}}(q) &= \int d^4x\, e^{ip\cdot x} \int d^4y\, e^{-iq\cdot x}\overbrace{\psi(x)\bar{\psi}}(y) \\
&= \int d^4x\, e^{ip\cdot x} \int d^4y\, e^{-iq\cdot y} \int \frac{d^4k}{(2\pi)^4}\frac{i\not{k}}{k^2}e^{-ik\cdot(x-y)} \\
&= \frac{i\not{p}}{p^2}(2\pi)^4\,\delta^{(4)}(p-q).
\end{aligned}$$

(4.47) より e^{-ipx} は位置 x に運動量 p が入るものとする (e.g. (4.47), p. 507).

[*1] $\bar{\psi}(p)$ は $\psi(p)$ の Hermite 共役に対し右から γ^0 をかけたもの

Chapter 2

The Klein-Gordon Field

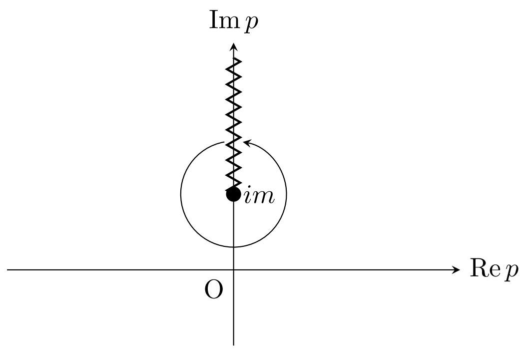
2.4 The Klein-Gordon Field in spacetime

(2.52)

積分

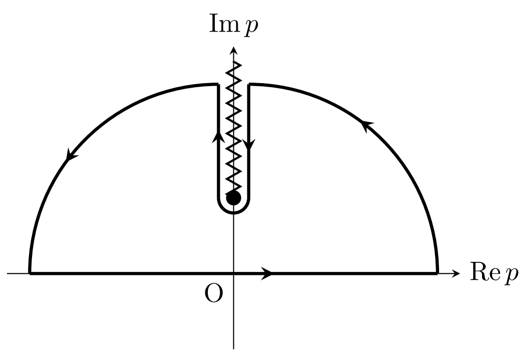
$$\int_{-\infty}^{\infty} dp \, \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

を計算する．複素平面では，$1/\sqrt{p^2+m^2}$ は $p=\pm im$ に極をもち，$[\pm im, \pm\infty)$ を載線 (branch cut) に取れる．



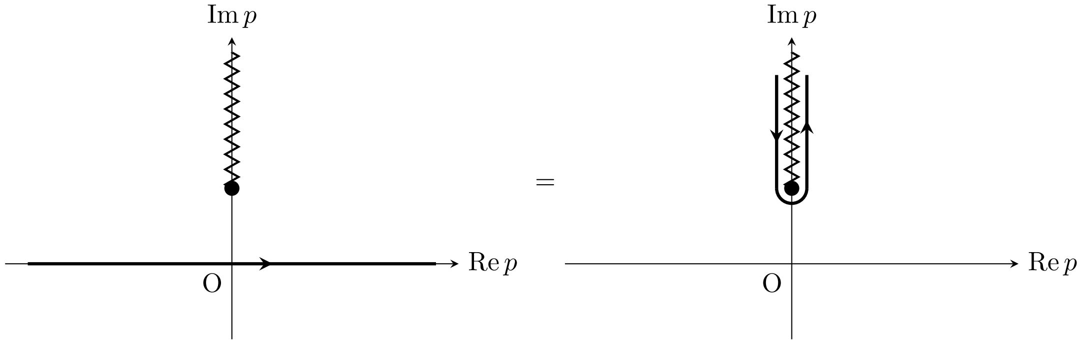
branch cut の左右で p^2+m^2 の偏角は 2π 異なるので，$\sqrt{p^2+m^2}$ の偏角は π 異なる．すなわち，左右で被積分函数の符号は入れ替わる．

次の経路で，積分値は 0.



(2.52)

大きい円弧に沿った積分は 0 なので，結局，



左側と右側の積分は逆向きで，被積分函数の符号が逆なので，積分値は等しい．従って，$p = i\rho$ とすれば，

$$\int_{-\infty}^{\infty} = 2\int_{im}^{i\infty} dp = 2i\int_{m}^{\infty} d\rho$$

となる．

Chapter 3

The Dirac Field

Problems

Problem 3.4: The Quantized Majorana Field

Majorana フェルミオンのモード展開は次式で与えられる：

$$\chi = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{(p\sigma)}{2E_{\boldsymbol{p}}}} \sum_s \left(a^s_{\boldsymbol{p}} \xi^s e^{-ipx} - i\sigma^2 a^{s\dagger}_{\boldsymbol{p}} \xi^s e^{ipx} \right),$$

$$\chi^\dagger = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\boldsymbol{p}}}} \sum_s \left(a^{s\dagger}_{\boldsymbol{p}} \xi^{s\dagger} e^{ipx} + i a^s_{\boldsymbol{p}} \xi^{s\dagger} \sigma^2 e^{-ipx} \right) \sqrt{(p\sigma)},$$

$$\chi^* = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{(p\sigma^*)}{2E_{\boldsymbol{p}}}} \sum_s \left(a^{s\dagger}_{\boldsymbol{p}} \xi^s e^{ipx} - i\sigma^2 a^s_{\boldsymbol{p}} \xi^s e^{-ipx} \right),$$

$$\chi^\top = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\boldsymbol{p}}}} \sum_s \left(a^s_{\boldsymbol{p}} \xi^{s\dagger} e^{-ipx} + i a^{s\dagger}_{\boldsymbol{p}} \xi^{s\dagger} \sigma^2 e^{ipx} \right) \sqrt{(p\sigma^\top)},$$

$$(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\chi = \int \frac{d^3p}{(2\pi)^3} i(\boldsymbol{p} \cdot \boldsymbol{\sigma}) \sqrt{\frac{(p\sigma)}{2E_{\boldsymbol{p}}}} \sum_s \left(a^s_{\boldsymbol{p}} \xi^s e^{-ipx} + i\sigma^2 a^{s\dagger}_{\boldsymbol{p}} \xi^s e^{ipx} \right).$$

ハミルトニアンは

$$H_{\text{Majorana}} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}} \dot{\chi} - \mathcal{L} \right) = \int d^3x \left[i\chi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \chi + \frac{im}{2} (\chi^\dagger \sigma^2 \chi^* - \chi^\top \sigma^2 \chi) \right]. \qquad [3.0.1]$$

[3.0.1] 第 1 項（$e^{\pm p_0 t}$ は省略）は

$$\begin{aligned}
&\int d^3x \, i\chi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \chi \\
&= \int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\boldsymbol{p}} 2E_{\boldsymbol{q}}}} \sum_{r,s} \left(a^{r\dagger}_{\boldsymbol{p}} \xi^{r\dagger} e^{ipx} + i a^r_{\boldsymbol{p}} \xi^{r\dagger} \sigma^2 e^{-ipx} \right) \\
&\qquad \times \sqrt{(p\sigma)} (\boldsymbol{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} \left(a^s_{\boldsymbol{q}} \xi^s e^{-iqx} + i\sigma^2 a^{s\dagger}_{\boldsymbol{q}} \xi^s e^{iqx} \right) \\
&= \int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\boldsymbol{p}} 2E_{\boldsymbol{q}}}} \sum_{r,s} \left(a^{r\dagger}_{\boldsymbol{p}} \xi^{r\dagger} e^{ipx} \right) \sqrt{(p\sigma)} (\boldsymbol{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} \left(a^s_{\boldsymbol{q}} \xi^s e^{-iqx} \right) \\
&\quad + \int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\boldsymbol{p}} 2E_{\boldsymbol{q}}}} \sum_{r,s} \left(a^{r\dagger}_{\boldsymbol{p}} \xi^{r\dagger} e^{ipx} \right) \sqrt{(p\sigma)} (\boldsymbol{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} \left(i\sigma^2 a^{s\dagger}_{\boldsymbol{q}} \xi^s e^{iqx} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}} \sum_{r,s} \left(ia^r_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{-ipx}\right) \sqrt{(p\sigma)}(\boldsymbol{q}\cdot\boldsymbol{\sigma})\sqrt{(q\sigma)} \left(a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right) \\
&+\int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}} \sum_{r,s} \left(ia^r_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{-ipx}\right) \sqrt{(p\sigma)}(\boldsymbol{q}\cdot\boldsymbol{\sigma})\sqrt{(q\sigma)} \left(i\sigma^2 a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right) \\
=&\int \frac{d^3p}{(2\pi)^3}\frac{-1}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} \times \xi^{r\dagger}\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\sigma)}\xi^s \\
&+\int \frac{d^3p}{(2\pi)^3}\frac{i}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r\dagger}_{\boldsymbol{p}} a^{s\dagger}_{-\boldsymbol{p}} \times \xi^{r\dagger}\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\bar\sigma)}\sigma^2\xi^s \\
&+\int \frac{d^3p}{(2\pi)^3}\frac{i}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r}_{\boldsymbol{p}} a^{s}_{-\boldsymbol{p}} \times \xi^{r\dagger}\sigma^2\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\bar\sigma)}\xi^s \\
&+\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{p}} \times \xi^{r\dagger}\sigma^2\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\sigma)}\sigma^2\xi^s
\end{aligned}
$$

である．$\sqrt{(p\sigma)}$ などを明示的に書くと

$$
\sqrt{(p\sigma)} = \frac{E_{\boldsymbol{p}}+m-\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2(E_{\boldsymbol{p}}+m)}}, \quad \sqrt{(p\bar\sigma)} = \frac{E_{\boldsymbol{p}}+m+\boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2(E_{\boldsymbol{p}}+m)}}
$$

となるので,

$$
\begin{aligned}
\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\sigma)} &= E_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{\sigma}) - |\boldsymbol{p}|^2, \\
\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\bar\sigma)}\sigma^2 &= m(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sigma^2, \\
\sigma^2\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\bar\sigma)} &= m\sigma^2(\boldsymbol{p}\cdot\boldsymbol{\sigma}), \\
\sigma^2\sqrt{(p\sigma)}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sqrt{(p\sigma)}\sigma^2 &= -E_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{\sigma}^*) - |\boldsymbol{p}|^2.
\end{aligned}
$$

従って，第 1 項を引き続き計算して

$$
= \int \frac{d^3p}{(2\pi)^3}\frac{-1}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r\dagger}_{\boldsymbol{p}} a^s_{\boldsymbol{p}} \times \xi^{r\dagger}\left[E_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{\sigma}) - |\boldsymbol{p}|^2\right]\xi^s \tag{3.0.2}
$$

$$
+ \int \frac{d^3p}{(2\pi)^3}\frac{i}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r\dagger}_{\boldsymbol{p}} a^{s\dagger}_{-\boldsymbol{p}} \times \xi^{r\dagger}\left[m(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sigma^2\right]\xi^s \tag{3.0.3}
$$

$$
+ \int \frac{d^3p}{(2\pi)^3}\frac{i}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r}_{\boldsymbol{p}} a^{s}_{-\boldsymbol{p}} \times \xi^{r\dagger}\left[m\sigma^2(\boldsymbol{p}\cdot\boldsymbol{\sigma})\right]\xi^s \tag{3.0.4}
$$

$$
+ \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a^{r}_{\boldsymbol{p}} a^{s\dagger}_{\boldsymbol{p}} \times \xi^{r\dagger}\left[-E_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{\sigma}^*) - |\boldsymbol{p}|^2\right]\xi^s \tag{3.0.5}
$$

となる．

[3.0.1] の第 2 項．

$$
\begin{aligned}
&\frac{im}{2}\int d^3x\, \chi^\dagger\sigma^2\chi^* \\
&= \frac{im}{2}\int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}} \sum_{r,s}\left(a^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}e^{ipx} + ia^r_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{-ipx}\right) \\
&\qquad \times \sqrt{(p\sigma)}\sigma^2\sqrt{(q\sigma^*)}\left(a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx} - i\sigma^2 a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}e^{ipx} + ia^r_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{-ipx}\right)\\
&\qquad \times \sqrt{(p\sigma)}\sqrt{(q\bar{\sigma})}\sigma^2\left(a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx} - i\sigma^2 a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right)\\
&= \frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}e^{ipx}\right)\sqrt{(p\sigma)}\sqrt{(q\bar{\sigma})}\sigma^2\left(a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right)\\
&\quad + \frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}e^{ipx}\right)\sqrt{(p\sigma)}\sqrt{(q\bar{\sigma})}\sigma^2\left(-i\sigma^2 a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right)\\
&\quad + \frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(ia^r_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{-ipx}\right)\sqrt{(p\sigma)}\sqrt{(q\bar{\sigma})}\sigma^2\left(a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right)\\
&\quad + \frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(ia^r_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{-ipx}\right)\sqrt{(p\sigma)}\sqrt{(q\bar{\sigma})}\sigma^2\left(-i\sigma^2 a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right)\\
&= \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^{s\dagger}_{-\boldsymbol{p}} \times \xi^{r\dagger}\sqrt{(p\sigma)}\sqrt{(p\sigma)}\sigma^2\xi^s\\
&\quad + \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{-i}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}} \times \xi^{r\dagger}\sqrt{(p\sigma)}\sqrt{(p\bar{\sigma})}\xi^s\\
&\quad + \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{i}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^{s\dagger}_{\boldsymbol{p}} \times \xi^{r\dagger}\sigma^2\sqrt{(p\sigma)}\sqrt{(p\bar{\sigma})}\sigma^2\xi^s\\
&\quad + \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^s_{-\boldsymbol{p}} \times \xi^{r\dagger}\sigma^2\sqrt{(p\sigma)}\sqrt{(p\sigma)}\xi^s
\end{aligned}
$$

に $\sqrt{(p\sigma)}\sqrt{(p\bar{\sigma})} = m$ などを代入して，

$$
= \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^{s\dagger}_{-\boldsymbol{p}} \times \xi^{r\dagger}\left[(p\sigma)\sigma^2\right]\xi^s \tag{3.0.6}
$$

$$
+ \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{-im}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}} \times \xi^{r\dagger}\xi^s \tag{3.0.7}
$$

$$
+ \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{im}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^{s\dagger}_{\boldsymbol{p}} \times \xi^{r\dagger}\xi^s \tag{3.0.8}
$$

$$
+ \frac{im}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^s_{-\boldsymbol{p}} \times \xi^{r\dagger}\sigma^2(p\sigma)\xi^s \tag{3.0.9}
$$

を得る．

[3.0.1] の第 3 項．

$$
\begin{aligned}
&-\frac{im}{2}\int d^3x\,\chi^\top\sigma^2\chi\\
&= -\frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^r_{\boldsymbol{p}}\xi^{r\dagger}e^{-ipx} + ia^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{ipx}\right)\\
&\qquad \times \sqrt{(p\sigma^\top)}\sigma^2\sqrt{(q\sigma)}\left(a^s_{\boldsymbol{q}}\xi^s e^{-iqx} - i\sigma^2 a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right)\\
&= -\frac{im}{2}\int d^3x \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^r_{\boldsymbol{p}}\xi^{r\dagger}e^{-ipx} + ia^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{ipx}\right)\\
&\qquad \times \sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(q\sigma)}\left(a^s_{\boldsymbol{q}}\xi^s e^{-iqx} - i\sigma^2 a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{im}{2}\int d^3x\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^r_{\boldsymbol{p}}\xi^{r\dagger}e^{-ipx}\right)\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(q\sigma)}\left(a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right)\\
&\quad -\frac{im}{2}\int d^3x\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(a^r_{\boldsymbol{p}}\xi^{r\dagger}e^{-ipx}\right)\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(q\sigma)}\left(-i\sigma^2 a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right)\\
&\quad -\frac{im}{2}\int d^3x\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(ia^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{ipx}\right)\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(q\sigma)}\left(a^s_{\boldsymbol{q}}\xi^s e^{-iqx}\right)\\
&\quad -\frac{im}{2}\int d^3x\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{r,s}\left(ia^{r\dagger}_{\boldsymbol{p}}\xi^{r\dagger}\sigma^2 e^{ipx}\right)\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(q\sigma)}\left(-i\sigma^2 a^{s\dagger}_{\boldsymbol{q}}\xi^s e^{iqx}\right)\\
&= -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^s_{-\boldsymbol{p}}\times\xi^{r\dagger}\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(p\bar{\sigma})}\xi^s\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{-i}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^{s\dagger}_{\boldsymbol{p}}\times\xi^{r\dagger}\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(p\sigma)}\sigma^2\xi^s\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{i}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\sigma^2\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(p\sigma)}\xi^s\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^{s\dagger}_{-\boldsymbol{p}}\times\xi^{r\dagger}\sigma^2\sigma^2\sqrt{(p\bar{\sigma})}\sqrt{(p\bar{\sigma})}\sigma^2\xi^s
\end{aligned}
$$

となるので,

$$
\begin{aligned}
&= -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^s_{-\boldsymbol{p}}\times\xi^{r\dagger}\left[\sigma^2(p\bar{\sigma})\right]\xi^s && [3.0.10]\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{-im}{2E_{\boldsymbol{p}}}\sum_{r,s}a^r_{\boldsymbol{p}}a^{s\dagger}_{\boldsymbol{p}}\times\xi^{r\dagger}\xi^s && [3.0.11]\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{im}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\xi^s && [3.0.12]\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^{s\dagger}_{-\boldsymbol{p}}\times\xi^{r\dagger}\left[(p\bar{\sigma})\sigma^2\right]\xi^s && [3.0.13]
\end{aligned}
$$

を得る.

$a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}$ の項は次のようになる：

$$
\begin{aligned}
&[3.0.2]+[3.0.7]+[3.0.12]\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{-1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\left[E_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{\sigma})-|\boldsymbol{p}|^2\right]\xi^s\\
&\quad +\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{-im}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\xi^s\\
&\quad -\frac{im}{2}\int\frac{d^3p}{(2\pi)^3}\frac{im}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\xi^s\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\left[-E_{\boldsymbol{p}}(\boldsymbol{p}\cdot\boldsymbol{\sigma})+|\boldsymbol{p}|^2\right]\xi^s+\int\frac{d^3p}{(2\pi)^3}\frac{m^2}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\xi^s\\
&= \int\frac{d^3p}{(2\pi)^3}\frac{|\boldsymbol{p}|^2+m^2}{2E_{\boldsymbol{p}}}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}\xi^s-\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s}a^{r\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}}\times\xi^{r\dagger}(\boldsymbol{p}\cdot\boldsymbol{\sigma})\xi^s
\end{aligned}
$$

$$
\begin{aligned}
&= \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \sum_{r,s} a_{\boldsymbol{p}}^{r\dagger} a_{\boldsymbol{p}}^{s} \delta^{rs} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_{\boldsymbol{p}}^{r\dagger} a_{\boldsymbol{p}}^{s} \times \xi^{r\dagger} (\boldsymbol{p}\cdot\boldsymbol{\sigma}) \xi^s \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \sum_{s} a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^{s}
\end{aligned} \qquad [3.0.14]
$$

（最後の式変形では被積分函数が奇函数であることを使った）．$a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger}$ の項は次のようになる：

$$
\begin{aligned}
&[3.0.5] + [3.0.8] + [3.0.11] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \left[-E_{\boldsymbol{p}} (\boldsymbol{p}\cdot\boldsymbol{\sigma}^*) - |\boldsymbol{p}|^2 \right] \xi^s \\
&\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{im}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \\
&- \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-im}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \left[-E_{\boldsymbol{p}} (\boldsymbol{p}\cdot\boldsymbol{\sigma}^*) - |\boldsymbol{p}|^2 \right] \xi^s - \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \\
&= - \int \frac{d^3p}{(2\pi)^3} \frac{|\boldsymbol{p}|^2 + m^2}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} (\boldsymbol{p}\cdot\boldsymbol{\sigma}^*) \xi^s \\
&= - \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \delta^{rs} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} (\boldsymbol{p}\cdot\boldsymbol{\sigma}^*) \xi^s \\
&= - \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \sum_{s} a_{\boldsymbol{p}}^{s} a_{\boldsymbol{p}}^{s\dagger}
\end{aligned} \qquad [3.0.15]
$$

（最後の式変形では被積分函数が奇函数であることを使った）．$a_{\boldsymbol{p}}^{r\dagger} a_{-\boldsymbol{p}}^{s\dagger}$ の項は次のようになる：

$$
\begin{aligned}
&[3.0.3] + [3.0.6] + [3.0.13] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r\dagger} a_{-\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \left[m (\boldsymbol{p}\cdot\boldsymbol{\sigma}) \sigma^2 \right] \xi^s \\
&\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r\dagger} a_{-\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \left[(p\sigma) \sigma^2 \right] \xi^s \\
&\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r\dagger} a_{-\boldsymbol{p}}^{s\dagger} \times \xi^{r\dagger} \left[(p\bar{\sigma}) \sigma^2 \right] \xi^s \\
&= 0.
\end{aligned} \qquad [3.0.16]
$$

$a_{\boldsymbol{p}}^{r} a_{-\boldsymbol{p}}^{s}$ の項は次のようになる：

$$
\begin{aligned}
&[3.0.4] + [3.0.9] + [3.0.10] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{-\boldsymbol{p}}^{s} \times \xi^{r\dagger} \left[m \sigma^2 (\boldsymbol{p}\cdot\boldsymbol{\sigma}) \right] \xi^s \\
&\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{-\boldsymbol{p}}^{s} \times \xi^{r\dagger} \sigma^2 (p\sigma) \xi^s \\
&\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \sum_{r,s} a_{\boldsymbol{p}}^{r} a_{-\boldsymbol{p}}^{s} \times \xi^{r\dagger} \left[\sigma^2 (p\bar{\sigma}) \right] \xi^s
\end{aligned}
$$

$$= 0. \tag{3.0.17}$$

[3.0.14][3.0.15][3.0.16][3.0.17] から，

$$\begin{aligned} H_{\text{Majorana}} &= \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \sum_s a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s - \int \frac{d^3p}{(2\pi)^3} \frac{E_{\boldsymbol{p}}}{2} \sum_s a_{\boldsymbol{p}}^s a_{\boldsymbol{p}}^{s\dagger} \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \sum_s a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s \end{aligned}$$

となる（最後の計算では，発散する定数を無視した）．これは Dirac 場のハミルトニアン

$$H_{\text{Dirac}} = \int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \sum_s \left(a_{\boldsymbol{p}}^{s\dagger} a_{\boldsymbol{p}}^s + b_{\boldsymbol{p}}^{s\dagger} b_{\boldsymbol{p}}^s \right)$$

の半分である．

Chapter 4

Interacting Fields and Feynman Diagrams

Problems

Problem 4.3: Linear sigma model

(d)

ポテンシャルは

$$V = -\frac{1}{2}\mu^2 \boldsymbol{\Phi}\cdot\boldsymbol{\Phi} + \frac{\lambda}{4}(\boldsymbol{\Phi}\cdot\boldsymbol{\Phi})^2 - a\Phi^N$$

で与えられる．V が $\Phi^i = 0$ で極小となる v を求める．

$$\frac{\partial V}{\partial \Phi^i} = (-\mu^2 + \lambda\boldsymbol{\Phi}\cdot\boldsymbol{\Phi})\Phi^i - a\delta^{iN}$$

に $\Phi^i = 0$ $(1 \le i \le N-1)$, $\Phi^N = v$ を代入して,

$$(-\mu^2 + \lambda v^2)v\delta^{iN} - a\delta^{iN} = 0.$$

a は十分小さいので,

$$v = \frac{\mu}{\sqrt{\lambda}} + \frac{a}{2\mu^2}$$

であり,

$$\Phi^N = \frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2}.$$

V の表式は

$$\begin{aligned}
V &= -\frac{1}{2}\mu^2 \boldsymbol{\Phi}\cdot\boldsymbol{\Phi} + \frac{\lambda}{4}(\boldsymbol{\Phi}\cdot\boldsymbol{\Phi})^2 - a\Phi^N \\
&= -\frac{\mu^2}{2}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2}\right)^2\right\} + \frac{\lambda}{4}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2}\right)^2\right\}^2 - a\left(\frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2}\right) \\
&\simeq -\frac{\mu^2}{2}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2 + 2\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)\frac{a}{2\mu^2}\right\} \\
&\quad + \frac{\lambda}{4}\left[\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right\}^2 + 2\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right\}2\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)\frac{a}{2\mu^2}\right] \\
&\quad - a\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right).
\end{aligned}$$

a を含まない項を先に計算する（これは (b) で計算した）：

$$
\begin{aligned}
V_0 &= -\frac{\mu^2}{2}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right\} + \frac{\lambda}{4}\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right\}^2 \\
&= -\frac{\mu^2}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) - \frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) + \frac{\lambda}{4}\left\{(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) + \left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right)\right\}^2 \\
&= -\frac{\mu^2}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) - \frac{\mu^2}{2}\left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) \\
&\quad + \frac{\lambda}{4}\left\{(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) + 2(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right) + \left(\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right)^2\right\} \\
&= -\frac{\mu^2}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) - \frac{\mu^4}{2\lambda} - \frac{\mu^3}{\sqrt{\lambda}}\sigma - \frac{\mu^2}{2}\sigma^2 \\
&\quad + \frac{\lambda}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 + \frac{\mu^2}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) + \sqrt{\lambda}\mu(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma + \frac{\lambda}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2 \\
&\quad + \frac{\mu^4}{4\lambda} + \mu^2\sigma^2 + \frac{\lambda}{4}\sigma^4 + \frac{\mu^3}{\sqrt{\lambda}}\sigma + \frac{1}{2}\mu^2\sigma^2 + \sqrt{\lambda}\mu\sigma^3 \\
&= -\frac{\mu^4}{4\lambda} + \sqrt{\lambda}\mu\sigma^3 + \mu^2\sigma^2 + \frac{\lambda}{4}\sigma^4 + \frac{\lambda}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 + \sqrt{\lambda}\mu(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma + \frac{\lambda}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2.
\end{aligned}
$$

次に，a を含む項を計算する：

$$
\begin{aligned}
V_a &= -\mu^2\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)\frac{a}{2\mu^2} + \lambda\left\{\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)^2\right\}\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)\frac{a}{2\mu^2} - a\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right) \\
&= a\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)\left\{-\frac{3}{2} + \frac{\lambda}{2\mu^2}\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right)\right\} \\
&= a\left(\frac{\mu}{\sqrt{\lambda}} + \sigma\right)\frac{\lambda}{2\mu^2}\left(\boldsymbol{\pi}\cdot\boldsymbol{\pi} - 2\frac{\mu^2}{\lambda} + 2\frac{\mu}{\sqrt{\lambda}}\sigma + \sigma^2\right).
\end{aligned}
$$

以上から

$$
\begin{aligned}
V &= V_0 + V_a \\
&= \frac{1}{2}\left(2\mu^2 + \frac{3a\sqrt{\lambda}}{\mu}\right)\sigma^2 + \frac{1}{2}\frac{a\sqrt{\lambda}}{\mu}(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) \\
&\quad + \left(\sqrt{\lambda}\mu + \frac{a\lambda}{2\mu^2}\right)(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma + \left(\sqrt{\lambda}\mu + \frac{a\lambda}{2\mu^2}\right)\sigma^3 + \frac{\lambda}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 + \frac{\lambda}{2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2 + \frac{\lambda}{4}\sigma^4 \\
&\quad + \text{const.}
\end{aligned}
$$

質量は

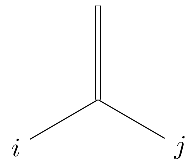
$$
{m_\sigma}^2 = 2\mu^2 + \frac{3a\sqrt{\lambda}}{\mu}, \quad {m_\pi}^2 = \frac{a\sqrt{\lambda}}{\mu}.
$$

propagator は

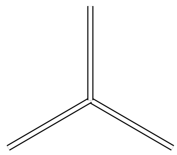
$$
= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - {m_\sigma}^2 + i\epsilon}e^{-ip(x-y)},
$$

$$
= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - {m_\pi}^2 + i\epsilon}e^{-ip(x-y)}\delta^{ij}.
$$

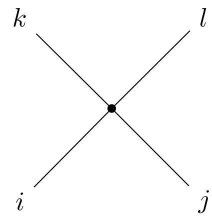
vertex factor は



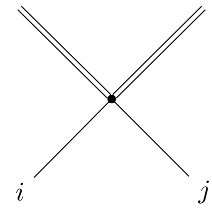
$$= -2i\left(\sqrt{\lambda}\mu + \frac{a\lambda}{2\mu^2}\right)\delta^{ij}$$



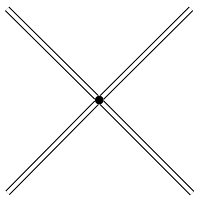
$$= -6i\left(\sqrt{\lambda}\mu + \frac{a\lambda}{2\mu^2}\right)$$



$$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$$



$$= -2i\lambda\delta^{ij}$$



$$= -6i\lambda$$

で与えられる．

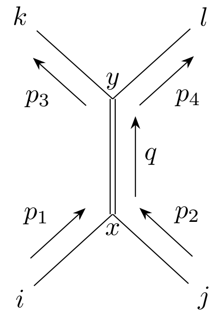
T 行列要素

$$T = \left\langle p_3^k p_4^l \middle| T\exp\left(-\int d^4x\,\mathcal{H}_{\text{int}}\right) \middle| p_1^i p_2^j \right\rangle$$

を計算する．

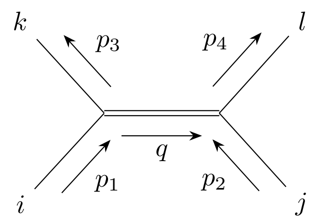
まず，2 次の展開を考える：

$$\sum_{m,n}\left\langle 0 \middle| a^k_{\boldsymbol{p_3}} a^l_{\boldsymbol{p_4}} \frac{(-i)^2}{2!}(\sqrt{\lambda}\mu)^2 \int d^4x d^4y\, N\{\pi^m(y)\pi^m(y)\sigma(y)\pi^n(x)\pi^n(x)\sigma(x)\} a^{i\dagger}_{\boldsymbol{p_1}} a^{j\dagger}_{\boldsymbol{p_2}} \middle| 0 \right\rangle.$$



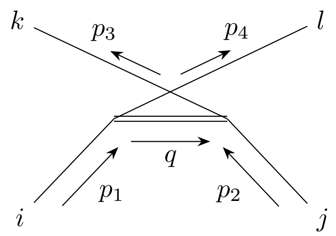
上図に対応する項は 4 通り存在し，x と y の交換を考慮に入れて，

$$
\begin{aligned}
&-4\lambda\mu^2 \int d^4x d^4y\, e^{i(p_3+p_4)y} \int \frac{d^4q}{(2\pi^4)} \frac{ie^{iq(y-x)}}{q^2 - {m_\sigma}^2} e^{-i(p_1+p_2)x} \delta^{ij}\delta^{kl} \\
&= -4\lambda\mu^2 \int \frac{d^4q}{(2\pi^4)} \frac{1}{q^2 - {m_\sigma}^2} \int d^4y\, e^{i(p_3+p_4-q)y} \int d^4x\, e^{-i(p_1+p_2-q)x} \delta^{ij}\delta^{kl} \\
&= -(2\pi)^4 4i\lambda\mu^2 \int \frac{d^4q}{q^2 - {m_\sigma}^2}\, \delta^{(4)}(p_3+p_4-q)\, \delta^{(4)}(p_1+p_2-q) \delta^{ij}\delta^{kl} \\
&= \frac{-4\lambda\mu^2}{(p_1+p_2)^2 - {m_\sigma}^2} \delta^{ij}\delta^{kl} i(2\pi)^4\, \delta^{(4)}(p_1+p_2-p_3-p_4).
\end{aligned}
$$



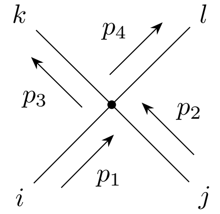
この場合は

$$
\frac{-4\lambda\mu^2}{(p_1-p_3)^2 - {m_\sigma}^2} \delta^{ik}\delta^{jl} i(2\pi)^4\, \delta^{(4)}(p_1+p_2-p_3-p_4).
$$



この場合は

$$
\frac{-4\lambda\mu^2}{(p_1-p_4)^2 - {m_\sigma}^2} \delta^{il}\delta^{jk} i(2\pi)^4\, \delta^{(4)}(p_1+p_2-p_3-p_4).
$$



この場合は

$$-2i\lambda\left(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk}+\delta^{ik}\delta^{jl}\right)i(2\pi)^4\,\delta^{(4)}(p_1+p_2-p_3-p_4).$$

ΔV によって ${m_\sigma}^2 \neq 2\mu^2$ となったので，$p_i \to 0$ の極限でも，これらの和は 0 とならない.

Chapter 5

Elementary Processes of Quantum Electrodynamics

5.2 $e^+e^- \to \mu^+\mu^-$: Helicity Structure

(5.28)

Dirac 方程式の解は (A.19) で与えられる：

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}.$$

高エネルギー極限では (A.20) のように，

$$u^s(p) \approx \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\xi^s \\ \frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\xi^s \end{pmatrix}, \quad v^s(p) \approx \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\eta^s \\ -\frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\eta^s \end{pmatrix}.$$

電子の spinor は $\xi = {}^\top(1,0)$ が $+z$ ($\sigma^3\xi = +\xi$)．陽電子の spinor は $\eta = {}^\top(0,1)$ が $+z$（電子と逆）(p. 61)

2 成分の spinor ξ が $(\hat{p}\cdot\boldsymbol{\sigma})\xi = +\xi$ を満たすとき helicity を右と定義する．陽電子の場合は spinor と粒子の spin が逆なので，helicity も逆になる (p. 142, 144)

電子は z 方向の spin 上向きなので，spinor は $\xi = {}^\top(1,0)$．$\hat{p} = (0,0,1)$ の向きに進むので，helicity は右．$\hat{p}\cdot\boldsymbol{\sigma} = \sigma^3$ なので，

$$u \approx \sqrt{2E}\begin{pmatrix} \frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\xi \\ \frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\xi \end{pmatrix} = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

陽電子は z 方向の粒子 spin が上向きなので，spinor は $\eta = {}^\top(0,1)$．$\hat{p} = (0,0,-1)$ の向きに進むので，粒

子 helicity は左．$\hat{p}\cdot\boldsymbol{\sigma} = -\sigma^3$ なので，

$$v \approx \sqrt{2E}\begin{pmatrix}\frac{1}{2}(1-\hat{p}\cdot\boldsymbol{\sigma})\eta \\ -\frac{1}{2}(1+\hat{p}\cdot\boldsymbol{\sigma})\eta\end{pmatrix} = \sqrt{2E}\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}.$$

5.5 Compton Scattering

(5.99)

入射電子は $-z$ の向きに進み，helicity は右とする．z 方向の spin 下向きなので

$$\hat{p} = (0,0,-1), \quad \xi = \begin{pmatrix}0\\1\end{pmatrix}, \quad u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

(5.97) が非零となるのは散乱電子 $u^\dagger(p')$ の第 3, 4 成分が非零，すなわち helicity が右の場合．さらに，電子は $+z$ 側に散乱される (Figure 5.6) ので，$\xi^\dagger = (1,0)$ である．

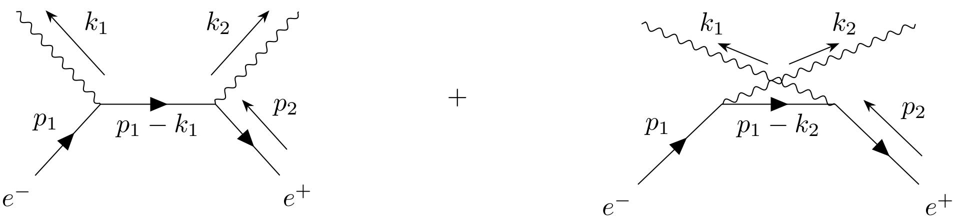
Problems

Problem 5.4: Positronium lifetime

[4] を簡略化した．

対消滅の不変振幅（0 次）

$e^- e^+ \to 2\gamma$ の過程を考える．



対消滅の不変振幅を運動量 p の 0 次までの精度で求める．

$$p_1^\mu = (E,0), \qquad p_2^\mu = (E,0), \qquad k_1^\mu = (E,\boldsymbol{k}), \qquad k_2^\mu = (E,-\boldsymbol{k}), \qquad |\boldsymbol{k}| = E.$$

光子の偏極は次のようにおく：

$$\epsilon_\pm^\mu(k_1) = \epsilon_{1\pm}^\mu = (0,\boldsymbol{\epsilon}_1), \qquad \boldsymbol{\epsilon}_1\cdot\boldsymbol{k}_1 = 0, \qquad \epsilon_\pm^\mu(k_2) = \epsilon_{2\pm}^\mu = (0,\boldsymbol{\epsilon}_2), \qquad \boldsymbol{\epsilon}_2\cdot\boldsymbol{k} = 0.$$

スピノルは次のように近似される：

$$u(p_1) = \begin{pmatrix}\sqrt{\sigma\cdot p_1}\xi \\ \sqrt{\bar\sigma\cdot p_1}\xi\end{pmatrix} \approx \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}, \qquad v(p_2) = \begin{pmatrix}\sqrt{\sigma\cdot p_2}\eta \\ -\sqrt{\bar\sigma\cdot p_2}\eta\end{pmatrix} \approx \sqrt{m}\begin{pmatrix}\eta \\ -\eta\end{pmatrix}. \tag{5.5.1}$$

さらに，

$$p_1 \cdot k_1 = E^2 \approx m^2, \qquad p_1 \cdot k_2 = E^2 \approx m^2, \qquad (p_1 \cdot k_1)(p_1 \cdot k_2) \approx m^4.$$

Dirac 方程式から次を得る：

$$\begin{aligned}(\not{p}_1 + m)\not{\epsilon}_1^* u(p_1) = 2(p_1 \cdot \epsilon_1^*)u(p_1) - \not{\epsilon}_1^*(\not{p}_1 - m)u(p_1) = 0,\\ (\not{p}_1 + m)\not{\epsilon}_2^* u(p_1) = 2(p_1 \cdot \epsilon_2^*)u(p_1) - \not{\epsilon}_2^*(\not{p}_1 - m)u(p_1) = 0.\end{aligned}$$

従って，不変振幅は

$$\begin{aligned}i\mathcal{M} &= -ie^2\overline{v}(p_2)\left[\not{\epsilon}_2^* \frac{\not{p}_1 - \not{k}_1 + m}{(p_1-k_1)^2 - m^2}\not{\epsilon}_1^* + \not{\epsilon}_1^* \frac{\not{p}_1 - \not{k}_2 + m}{(p_1-k_2)^2 - m^2}\not{\epsilon}_2^*\right]u(p_1)\\ &= -ie^2\overline{v}(p_2)\left[\not{\epsilon}_2^* \frac{-\not{p}_1 + \not{k}_1 - m}{2p_1 \cdot k_1}\not{\epsilon}_1^* + \not{\epsilon}_1^* \frac{-\not{p}_1 + \not{k}_2 - m}{2p_1 \cdot k_2}\not{\epsilon}_2^*\right]u(p_1)\\ &= -i\frac{e^2}{2}\overline{v}(p_2)\left[\frac{\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^*}{m^2} + \frac{\not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*}{m^2}\right]u(p_1)\\ &= -i\frac{e^2}{2m^2}\overline{v}(p_2)\left[\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^* + \not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*\right]u(p_1)\end{aligned} \tag{5.5.2}$$

となる．$[\cdots]$ を計算するが，まず次の式を示しておく：

$$\begin{aligned}\not{a}\not{b}\not{c} &= a_\mu b_\nu c_\rho \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \overline{\sigma}^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\rho \\ \overline{\sigma}^\rho & 0 \end{pmatrix}\\ &= a_\mu b_\nu c_\rho \begin{pmatrix} 0 & \sigma^\mu\overline{\sigma}^\nu\sigma^\rho \\ \overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho & 0 \end{pmatrix}\\ &= \begin{pmatrix} 0 & (a\cdot\sigma)(b\cdot\overline{\sigma})(c\cdot\sigma) \\ (a\cdot\overline{\sigma})(b\cdot\sigma)(c\cdot\overline{\sigma}) & 0 \end{pmatrix}.\end{aligned}$$

[5.5.2] の $[\cdots]$ のうち 3 つのガンマ行列を含む項は

$$\begin{aligned}\left[\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^* + \not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*\right] &= \begin{pmatrix} 0 & (\epsilon_2^*\cdot\sigma)(k_1\cdot\overline{\sigma})(\epsilon_1^*\cdot\sigma) \\ (\epsilon_2^*\cdot\overline{\sigma})(k_1\cdot\sigma)(\epsilon_1^*\cdot\overline{\sigma}) & 0 \end{pmatrix}\\ &\quad+ \begin{pmatrix} 0 & (\epsilon_1^*\cdot\sigma)(k_2\cdot\overline{\sigma})(\epsilon_2^*\cdot\sigma) \\ (\epsilon_1^*\cdot\overline{\sigma})(k_2\cdot\sigma)(\epsilon_2^*\cdot\overline{\sigma}) & 0 \end{pmatrix}\\ &= \begin{pmatrix} 0 & (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\overline{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) \\ (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\sigma)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) & 0 \end{pmatrix}\\ &\quad+ \begin{pmatrix} 0 & (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\overline{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) \\ (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\sigma)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) & 0 \end{pmatrix}\\ &=: \begin{pmatrix} 0 & m^2\overline{B}_+ \\ m^2 B_+ & 0 \end{pmatrix}.\end{aligned}$$

以上から

$$i\mathcal{M} = -i\frac{e^2}{2m^4}\overline{v}(p_2)\begin{pmatrix} 0 & -2m^2A + m^2\overline{B}_+ - (\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_- \\ 2m^2A + m^2B_+ - (\boldsymbol{p}\cdot\boldsymbol{k})B_- & 0 \end{pmatrix}u(p_1) \tag{5.5.3}$$

となる．各項の定義は

$$\begin{aligned}B_+ &= (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\sigma)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\sigma)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\ \overline{B}_+ &= (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\overline{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\overline{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}).\end{aligned} \tag{5.5.4}$$

$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{b}+i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ から

$$\begin{aligned}
(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) &= [\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{k}+i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})](\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})\\
&= i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})\\
&= i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})\cdot\boldsymbol{\epsilon}_2^*-\boldsymbol{\sigma}\cdot[(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})\times\boldsymbol{\epsilon}_2^*]\\
&= i(\boldsymbol{\epsilon}_2^*\times\boldsymbol{\epsilon}_1^*)\cdot\boldsymbol{k}-\boldsymbol{\sigma}\cdot[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\boldsymbol{k}-(\boldsymbol{k}\cdot\boldsymbol{\epsilon}_2^*)\boldsymbol{\epsilon}_1^*]\\
&= -i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}-(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\\
(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) &= i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}-(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)
\end{aligned}$$

となる．従って，

$$\begin{aligned}
\overline{B}_+-B_+ &= (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\overline{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})+(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\overline{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\
&\quad-(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\sigma)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})-(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\sigma)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\
&= (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot(\overline{\sigma}-\sigma))(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})+(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot(\overline{\sigma}-\sigma))(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\
&= -2(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})+2(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\
&= 4i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k},
\end{aligned} \qquad [5.5.5]$$

[5.5.3] に [5.5.1] を代入して，[5.5.5] を使えば，

$$\begin{aligned}
i\mathcal{M}(e_s^-e_r^+\to2\gamma) &= -i\frac{e^2}{2m^2}\overline{v}(p_2)\begin{pmatrix}0&\overline{B}_+\\B_+&0\end{pmatrix}u(p_1)\\
&= -i\frac{e^2}{2m}\begin{pmatrix}\eta^{r\dagger}&-\eta^{r\dagger}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\overline{B}_+\\B_+&0\end{pmatrix}\begin{pmatrix}\xi^s\\\xi^s\end{pmatrix}\\
&= i\frac{e^2}{2m}\eta^{r\dagger}(\overline{B}_+-B_+)\xi^s\\
&= -\frac{2e^2}{m}(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}(\eta^{r\dagger}\xi^s)
\end{aligned} \qquad [5.5.6]$$

となる．

ここで，各スピノルは (3.135)(3.136) から

$$\xi^\uparrow=\begin{pmatrix}1\\0\end{pmatrix},\qquad \xi^\downarrow=\begin{pmatrix}0\\1\end{pmatrix},\qquad \eta^\uparrow=\begin{pmatrix}0\\1\end{pmatrix},\qquad \eta^\downarrow=\begin{pmatrix}-1\\0\end{pmatrix} \qquad [5.5.7]$$

で与えられる．ラベルは全て電子・陽電子のスピンを表す．

S ポジトロニウムの構成

束縛状態の式 (5.43)

$$|B\rangle=\sqrt{2M}\int\frac{d^3k}{(2\pi)^3}\tilde{\psi}(\boldsymbol{k})\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\left|\boldsymbol{k}\uparrow,-\boldsymbol{k}\uparrow\right\rangle$$

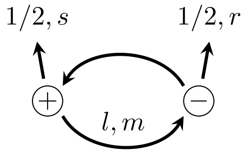
を角運動量を考慮した形に拡張する．すなわち，スピン S，軌道角運動量 l，全角運動量 J，全角運動量の射影 M の状態 $|^{2S+1}l_J;M\rangle$ を構成しよう[*1]．これには，スピンの射影が S_z で軌道角運動量の射影が $M-S_Z=:m$

[*1] スピンは粒子の内在的な量なので，電子のスピン，陽電子のスピンを足す．それに対し軌道角運動量は粒子の相対運動に起因するので，電子と陽電子について和を取ることは行わない

の状態を Clebash-Gordan 係数 $\langle lS; mS_z | lS; JM \rangle$ によって足せば良い：

$$|^{2S+1}l_J; M\rangle = \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{S_z=-S}^{S} \langle lS; mS_z | lS; JM \rangle \, \tilde{\psi}_{lm}(\boldsymbol{p}) \, |S, S_z\rangle_{\boldsymbol{p}} \tag{5.5.8}$$

（ただし，慣例に従い $l=0$ は S，$l=1$ は P などと表記する）．なお，電子の運動量は $\boldsymbol{p}$，陽電子の運動量は $-\boldsymbol{p}$ とする．換算質量は $m/2$ なので，相対運動量は $\boldsymbol{p}$ である．



（電子と陽電子の合計）スピン S，射影 S_z，相対運動量 $\boldsymbol{p}$ の状態 $|S, S_z\rangle_{\boldsymbol{p}}$ は，スピン射影 s で運動量 $\boldsymbol{p}$ の電子と，スピン射影 $r := S_z - s$ で運動量 $\boldsymbol{p}$ の陽電子の線形結合で表せる：

$$\begin{aligned}
|S, S_z\rangle_{\boldsymbol{p}} &= \sum_s \left\langle \frac{1}{2}\frac{1}{2}; sr \,\middle|\, \frac{1}{2}\frac{1}{2}; SS_z \right\rangle \left| \frac{1}{2}, s \right\rangle_{\boldsymbol{p}} \left| \frac{1}{2}, r \right\rangle_{-\boldsymbol{p}} \\
&= \sum_s \left\langle \frac{1}{2}\frac{1}{2}; sr \,\middle|\, \frac{1}{2}\frac{1}{2}; SS_z \right\rangle \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} a_{\boldsymbol{p}}^{s\dagger} b_{-\boldsymbol{p}}^{r\dagger} |0\rangle .
\end{aligned}$$

まず，S 状態 $(l=0)$ のポジトロニウムについて考える（$m=0$ なので $M = S_z$）．$|^1S_0; 0\rangle$ は $J = M = 0$, $S = S_z = 0$ なのでスピンは singlet：

$$|^1S_0; 0\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\boldsymbol{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} \frac{|\boldsymbol{p}\uparrow, -\boldsymbol{p}\downarrow\rangle - |\boldsymbol{p}\downarrow, -\boldsymbol{p}\uparrow\rangle}{\sqrt{2}}. \tag{5.5.9}$$

$|^3S_1; 0\rangle$ は $J=1$, $M=0$, $S=1$, $S_z=0$ なのでスピンは triplet：

$$|^3S_1; 0\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\boldsymbol{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} \frac{|\boldsymbol{p}\uparrow, -\boldsymbol{p}\downarrow\rangle + |\boldsymbol{p}\downarrow, -\boldsymbol{p}\uparrow\rangle}{\sqrt{2}}.$$

$|^3S_1; 1\rangle$ は $J=1$, $M=1$, $S=1$, $S_z=1$ なのでスピンは triplet：

$$|^3S_1; 1\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\boldsymbol{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |\boldsymbol{p}\uparrow, -\boldsymbol{p}\uparrow\rangle .$$

$|^3S_1; -1\rangle$ は $J=1$, $M=-1$, $S=1$, $S_z=-1$ なのでスピンは triplet：

$$|^3S_1; -1\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\boldsymbol{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |\boldsymbol{p}\downarrow, -\boldsymbol{p}\downarrow\rangle .$$

S ポジトロニウムの崩壊

不変振幅 $\mathcal{M}$ の定義 (4.73) に注意して，[5.5.6][5.5.7][5.5.9] から

$$
\begin{aligned}
i\mathcal{M}(^1S_0 \to 2\gamma) &= 2\sqrt{m}\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}_{00}(\boldsymbol{p})\frac{1}{2m}\frac{i\mathcal{M}(e^-_\uparrow e^+_\downarrow \to 2\gamma) - i\mathcal{M}(e^-_\downarrow e^+_\uparrow \to 2\gamma)}{\sqrt{2}} \\
&= 2\sqrt{m}\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}_{00}(\boldsymbol{p})\frac{1}{2m}\left(-\frac{2e^2}{m}\right)(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}\frac{\eta^{\downarrow\dagger}\xi^\uparrow - \eta^{\uparrow\dagger}\xi^\downarrow}{\sqrt{2}} \\
&= 2\sqrt{m}\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}_{00}(\boldsymbol{p})\frac{1}{2m}\left(-\frac{2e^2}{m}\right)(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}\frac{-2}{\sqrt{2}} \\
&= 2\sqrt{m}\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}_{00}(\boldsymbol{p})\frac{1}{2m}\left(-\frac{2e^2}{m}\right)(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}\frac{-2}{\sqrt{2}} \\
&= 2\sqrt{2}\frac{e^2}{m\sqrt{m}}(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}_{00}(\boldsymbol{p}) \\
&= 2\sqrt{2}\frac{e^2}{m\sqrt{m}}(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}\psi_{00}(0).
\end{aligned}
\tag{5.5.10}
$$

(A.26) から

$$
\sum_{\text{polarization}} \epsilon^{\mu *}\epsilon^\nu \to -g^{\mu\nu}
$$

なので，

$$
\begin{aligned}
|(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k}|^2 &= \sum_{\text{pol 1}}\sum_{\text{pol 2}}\sum_{ijklmn}\epsilon^{ijk}\epsilon_1^{i*}\epsilon_2^{j*}k^k\epsilon^{lmn}\epsilon_1^l\epsilon_2^m k^n \\
&\to \sum_{ijklmn}\epsilon^{ijk}\epsilon^{lmn}g^{il}g^{jm}k^k k^n \\
&= \sum_{klmn}\epsilon^{lmk}\epsilon^{lmn}k^k k^n \\
&= \sum_{lmn}\epsilon^{lmn}\epsilon^{lmn}k^n k^n \\
&= 2|\boldsymbol{k}|^2 = 2E^2 \approx 2m^2
\end{aligned}
\tag{5.5.11}
$$

を得る．

$n = 1$ の場合[*2]を考える．

$$
\psi_{100} = R_{10}(r)Y_{00}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\frac{2}{{a_0}^{3/2}}\exp\left(-\frac{r}{2a_0}\right)
$$

なので，

$$
|\psi_{100}(0)|^2 = \frac{m^3\alpha^3}{8\pi}. \tag{5.5.12}
$$

[5.5.10][5.5.11][5.5.12] から

$$
\sum_{\text{polarization}}|\mathcal{M}(1^1S_0 \to 2\gamma)|^2 = 8\frac{e^4}{m^3}2m^2\frac{m^3\alpha^3}{8\pi} = 32\pi m^2\alpha^5. \tag{5.5.13}
$$

[*2] 換算質量は $m/2$ なので，Bohr 半径 $a_0 = 2/\alpha m$

(4.86) から

$$\begin{aligned}
\Gamma(1^1S_0 \to 2\gamma) &= \frac{1}{2}\frac{1}{4m}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{4E_1E_2}|\mathcal{M}(^1S_0 \to 2\gamma)|^2(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2) \\
&= \frac{1}{8m}\int \frac{d^3k_1}{(2\pi)^2}\frac{1}{4|\boldsymbol{k}_1|^2}32\pi m^2\alpha^5\,\delta(2E_1-2m) \\
&= \pi m\alpha^5 4\pi\int \frac{d|\boldsymbol{k}_1|}{(2\pi)^2}\frac{1}{2}\,\delta(|\boldsymbol{k}_1|-m) \\
&= \frac{m\alpha^5}{2}
\end{aligned}$$

（出てくる光子は区別できないので，1/2 倍する）．

[5.5.10] と同様に考えれば，

$$\eta^{\uparrow\dagger}\xi^{\uparrow} = 0, \qquad \frac{\eta^{\uparrow\dagger}\xi^{\downarrow}+\eta^{\downarrow\dagger}\xi^{\uparrow}}{\sqrt{2}} = 0, \qquad \eta^{\downarrow\dagger}\xi^{\downarrow} = 0$$

なので，$\mathcal{M}(^3S_1 \to 2\gamma) = 0$ であることが分かる．すなわち，スピン 1 の 1^3S は 2 光子に崩壊しない．

対消滅の不変振幅（1 次）

対消滅の不変振幅を運動量 p の 1 次までの精度で求める．

$$p_1^\mu = (E,\boldsymbol{p}), \qquad p_2^\mu = (E,-\boldsymbol{p}), \qquad k_1^\mu = (E,\boldsymbol{k}), \qquad k_2^\mu = (E,-\boldsymbol{k}).$$

光子の偏極は次のようにおく：

$$\epsilon_\pm^\mu(k_1) = \epsilon_{1\pm}^\mu = (0,\boldsymbol{\epsilon}_1), \qquad \boldsymbol{\epsilon}_1\cdot\boldsymbol{k}_1 = 0, \qquad \epsilon_\pm^\mu(k_2) = \epsilon_{2\pm}^\mu = (0,\boldsymbol{\epsilon}_2), \qquad \boldsymbol{\epsilon}_2\cdot\boldsymbol{k}_2 = 0.$$

スピノルは次のように近似される：

$$u(p_1) = \begin{pmatrix}\sqrt{\sigma\cdot p_1}\xi \\ \sqrt{\bar\sigma\cdot p_1}\xi\end{pmatrix} \approx \sqrt{m}\begin{pmatrix}\left(1-\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\xi \\ \left(1+\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\xi\end{pmatrix}, \quad v(p_2) = \begin{pmatrix}\sqrt{\sigma\cdot p_2}\eta \\ -\sqrt{\bar\sigma\cdot p_2}\eta\end{pmatrix} \approx \sqrt{m}\begin{pmatrix}\left(1+\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\eta \\ -\left(1-\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\eta\end{pmatrix}. \tag{5.5.14}$$

さらに，

$$\begin{aligned}
p_1\cdot k_1 &= E^2 - \boldsymbol{p}\cdot\boldsymbol{k} \approx m^2 - \boldsymbol{p}\cdot\boldsymbol{k}, \\
p_1\cdot k_2 &= E^2 + \boldsymbol{p}\cdot\boldsymbol{k} \approx m^2 + \boldsymbol{p}\cdot\boldsymbol{k}, \\
(p_1\cdot k_1)(p_1\cdot k_2) &\approx m^4.
\end{aligned}$$

Dirac 方程式から次を得る：

$$\begin{aligned}
(\not{p}_1+m)\not{\epsilon}_1^* u(p_1) &= 2(p_1\cdot\epsilon_1^*)u(p_1) - \not{\epsilon}_1^*(\not{p}_1-m)u(p_1) = 2(p_1\cdot\epsilon_1^*)u(p_1), \\
(\not{p}_1+m)\not{\epsilon}_2^* u(p_1) &= 2(p_1\cdot\epsilon_2^*)u(p_1) - \not{\epsilon}_2^*(\not{p}_1-m)u(p_1) = 2(p_1\cdot\epsilon_2^*)u(p_1).
\end{aligned}$$

従って，不変振幅は

$$
\begin{aligned}
i\mathcal{M} &= -ie^2\overline{v}(p_2)\left[\not{\epsilon}_2^*\frac{\not{p}_1-\not{k}_1+m}{(p_1-k_1)^2-m^2}\not{\epsilon}_1^*+\not{\epsilon}_1^*\frac{\not{p}_1-\not{k}_2+m}{(p_1-k_2)^2-m^2}\not{\epsilon}_2^*\right]u(p_1)\\
&= -ie^2\overline{v}(p_2)\left[\not{\epsilon}_2^*\frac{-\not{p}_1+\not{k}_1-m}{2p_1\cdot k_1}\not{\epsilon}_1^*+\not{\epsilon}_1^*\frac{-\not{p}_1+\not{k}_2-m}{2p_1\cdot k_2}\not{\epsilon}_2^*\right]u(p_1)\\
&= -i\frac{e^2}{2}\overline{v}(p_2)\left[\frac{\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^*-2(p_1\cdot\epsilon_1^*)\not{\epsilon}_2^*}{m^2-\boldsymbol{p}\cdot\boldsymbol{k}}+\frac{\not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*-2(p_1\cdot\epsilon_2^*)\not{\epsilon}_1^*}{m^2+\boldsymbol{p}\cdot\boldsymbol{k}}\right]u(p_1)\\
&= -i\frac{e^2}{2m^4}\overline{v}(p_2)\left[(m^2+\boldsymbol{p}\cdot\boldsymbol{k})(\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^*-2(p_1\cdot\epsilon_1^*)\not{\epsilon}_2^*)+(m^2-\boldsymbol{p}\cdot\boldsymbol{k})(\not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*-2(p_1\cdot\epsilon_2^*)\not{\epsilon}_1^*)\right]u(p_1)
\end{aligned}
\tag{5.5.15}
$$

となる．$[\cdots]$ を計算するが，まず次の式を示しておく：

$$
\begin{aligned}
\not{a}\not{b}\not{c} &= a_\mu b_\nu c_\rho\begin{pmatrix}0 & \sigma^\mu\\ \overline{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\nu\\ \overline{\sigma}^\nu & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\rho\\ \overline{\sigma}^\rho & 0\end{pmatrix}\\
&= a_\mu b_\nu c_\rho\begin{pmatrix}0 & \sigma^\mu\overline{\sigma}^\nu\sigma^\rho\\ \overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & (a\cdot\sigma)(b\cdot\overline{\sigma})(c\cdot\sigma)\\ (a\cdot\overline{\sigma})(b\cdot\sigma)(c\cdot\overline{\sigma}) & 0\end{pmatrix}.
\end{aligned}
$$

[5.5.15] の $[\cdots]$ のうち m^2 と 1 つのガンマ行列を含む項は

$$
\begin{aligned}
&-2m^2(p_1\cdot\epsilon_1^*)\not{\epsilon}_2^*-2m^2(p_1\cdot\epsilon_2^*)\not{\epsilon}_1^*\\
&= -2m^2(p_1\cdot\epsilon_1^*)\begin{pmatrix}0 & \epsilon_2^*\cdot\sigma\\ \epsilon_2^*\cdot\overline{\sigma} & 0\end{pmatrix}-2m^2(p_1\cdot\epsilon_2^*)\begin{pmatrix}0 & \epsilon_1^*\cdot\sigma\\ \epsilon_1^*\cdot\overline{\sigma} & 0\end{pmatrix}\\
&= -2m^2(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)\begin{pmatrix}0 & \boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}\\ -\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma} & 0\end{pmatrix}-2m^2(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)\begin{pmatrix}0 & \boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}\\ -\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma} & 0\end{pmatrix}\\
&= -2m^2\begin{pmatrix}0 & (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})+(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\ -(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})-(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) & 0\end{pmatrix}\\
&=: \begin{pmatrix}0 & -2m^2A\\ 2m^2A & 0\end{pmatrix}.
\end{aligned}
$$

[5.5.15] の $[\cdots]$ のうち m^2 と 3 つのガンマ行列を含む項は

$$
\begin{aligned}
&m^2\left[\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^*+\not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*\right]\\
&\quad= m^2\begin{pmatrix}0 & (\epsilon_2^*\cdot\sigma)(k_1\cdot\overline{\sigma})(\epsilon_1^*\cdot\sigma)\\ (\epsilon_2^*\cdot\overline{\sigma})(k_1\cdot\sigma)(\epsilon_1^*\cdot\overline{\sigma}) & 0\end{pmatrix}\\
&\quad+ m^2\begin{pmatrix}0 & (\epsilon_1^*\cdot\sigma)(k_2\cdot\overline{\sigma})(\epsilon_2^*\cdot\sigma)\\ (\epsilon_1^*\cdot\overline{\sigma})(k_2\cdot\sigma)(\epsilon_2^*\cdot\overline{\sigma}) & 0\end{pmatrix}\\
&\quad= m^2\begin{pmatrix}0 & (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\overline{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})\\ (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\sigma)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) & 0\end{pmatrix}\\
&\quad+ m^2\begin{pmatrix}0 & (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\overline{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})\\ (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\sigma)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) & 0\end{pmatrix}\\
&=: \begin{pmatrix}0 & m^2\overline{B}_+\\ m^2B_+ & 0\end{pmatrix}.
\end{aligned}
$$

[5.5.15] の $[\cdots]$ のうち $\boldsymbol{p}\cdot\boldsymbol{k}$ と 3 つのガンマ行列を含む項は

$$
\boldsymbol{p}\cdot\boldsymbol{k}\left[\not{\epsilon}_2^*\not{k}_1\not{\epsilon}_1^*-\not{\epsilon}_1^*\not{k}_2\not{\epsilon}_2^*\right]
$$

$$
\begin{aligned}
&= \boldsymbol{p}\cdot\boldsymbol{k}\begin{pmatrix} 0 & (\epsilon_2^*\cdot\sigma)(k_1\cdot\overline{\sigma})(\epsilon_1^*\cdot\sigma) \\ (\epsilon_2^*\cdot\overline{\sigma})(k_1\cdot\sigma)(\epsilon_1^*\cdot\overline{\sigma}) & 0 \end{pmatrix} \\
&- \boldsymbol{p}\cdot\boldsymbol{k}\begin{pmatrix} 0 & (\epsilon_1^*\cdot\sigma)(k_2\cdot\overline{\sigma})(\epsilon_2^*\cdot\sigma) \\ (\epsilon_1^*\cdot\overline{\sigma})(k_2\cdot\sigma)(\epsilon_2^*\cdot\overline{\sigma}) & 0 \end{pmatrix} \\
&= \boldsymbol{p}\cdot\boldsymbol{k}\begin{pmatrix} 0 & (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\overline{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) \\ (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\sigma)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) & 0 \end{pmatrix} \\
&- \boldsymbol{p}\cdot\boldsymbol{k}\begin{pmatrix} 0 & (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\overline{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) \\ (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\sigma)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) & 0 \end{pmatrix} \\
&=: \begin{pmatrix} 0 & -(\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_- \\ -(\boldsymbol{p}\cdot\boldsymbol{k})B_- & 0 \end{pmatrix}.
\end{aligned}
$$

[5.5.15] の残りの項は運動量の 2 乗なので無視する：

$$
\begin{aligned}
&-2(\boldsymbol{p}\cdot\boldsymbol{k})(p_1\cdot\epsilon_1^*)\not{\epsilon}_2^* + 2(\boldsymbol{p}\cdot\boldsymbol{k})(p_1\cdot\epsilon_2^*)\not{\epsilon}_1^* \\
&= -2(\boldsymbol{p}\cdot\boldsymbol{k})(p_1\cdot\epsilon_1^*)\begin{pmatrix} 0 & \epsilon_2^*\cdot\sigma \\ \epsilon_2^*\cdot\overline{\sigma} & 0 \end{pmatrix} + 2(\boldsymbol{p}\cdot\boldsymbol{k})(p_1\cdot\epsilon_2^*)\begin{pmatrix} 0 & \epsilon_1^*\cdot\sigma \\ \epsilon_1^*\cdot\overline{\sigma} & 0 \end{pmatrix} \\
&= -2(\boldsymbol{p}\cdot\boldsymbol{k})(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)\begin{pmatrix} 0 & \boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma} \\ -\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} + 2(\boldsymbol{p}\cdot\boldsymbol{k})(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)\begin{pmatrix} 0 & \boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma} \\ -\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma} & 0 \end{pmatrix} \\
&= -2(\boldsymbol{p}\cdot\boldsymbol{k})\begin{pmatrix} 0 & (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) - (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) \\ -(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) & 0 \end{pmatrix} \\
&\approx 0.
\end{aligned}
$$

以上から

$$
i\mathcal{M} = -i\frac{e^2}{2m^4}\overline{v}(p_2)\begin{pmatrix} 0 & -2m^2A + m^2\overline{B}_+ - (\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_- \\ 2m^2A + m^2B_+ - (\boldsymbol{p}\cdot\boldsymbol{k})B_- & 0 \end{pmatrix} u(p_1) \tag{5.5.16}
$$

となる．各項の定義は

$$
\begin{aligned}
A &= (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) \\
B_\pm &= (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\sigma)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) \pm (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\sigma)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) \\
\overline{B}_\pm &= (\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(k_2\cdot\overline{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) \pm (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(k_1\cdot\overline{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}).
\end{aligned} \tag{5.5.17}
$$

$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ から

$$
\begin{aligned}
(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) &= [\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{k} + i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})](\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) \\
&= i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) \\
&= i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})\cdot\boldsymbol{\epsilon}_2^* - \boldsymbol{\sigma}\cdot[(\boldsymbol{\epsilon}_1^*\times\boldsymbol{k})\times\boldsymbol{\epsilon}_2^*] \\
&= i(\boldsymbol{\epsilon}_2^*\times\boldsymbol{\epsilon}_1^*)\cdot\boldsymbol{k} - \boldsymbol{\sigma}\cdot[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{\epsilon}_2^*)\boldsymbol{\epsilon}_1^*] \\
&= -i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k} - (\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*) \\
(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(\boldsymbol{k}\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) &= i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k} - (\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*) \\
(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) &= 2\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*
\end{aligned}
$$

となる．従って，

$$
\begin{aligned}
\overline{B}_+ - B_+ &= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot \overline{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot \overline{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&\quad - (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot \sigma)(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) - (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot \sigma)(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot (\overline{\sigma} - \sigma))(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot (\overline{\sigma} - \sigma))(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= -2(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + 2(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= 4i(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \boldsymbol{k}, \\
B_- - \overline{B}_- &= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot \sigma)(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) - (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot \sigma)(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&\quad - (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot \overline{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot \overline{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot (\sigma - \overline{\sigma}))(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) - (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot (\sigma - \overline{\sigma}))(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= 2(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + 2(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= -4(\boldsymbol{\sigma} \cdot \boldsymbol{k})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*), \\
B_+ + \overline{B}_+ &= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot \sigma)(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot \sigma)(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&\quad + (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(k_2 \cdot \overline{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(k_1 \cdot \overline{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= 2E(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + 2E(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&\approx 2m(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + 2m(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
&= 4m\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*.
\end{aligned}
\tag{5.5.18}
$$

[5.5.16] に [5.5.14] を代入して，[5.5.18] を使えば，

$$
\begin{aligned}
&i\mathcal{M}(e_s^- e_r^+ \to 2\gamma) \\
&= -i\frac{e^2}{2m^4}\overline{v}(p_2)\begin{pmatrix} 0 & -2m^2A + m^2\overline{B}_+ - (\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_- \\ 2m^2A + m^2B_+ - (\boldsymbol{p}\cdot\boldsymbol{k})B_- & 0 \end{pmatrix} u(p_1) \\
&= -i\frac{e^2}{2m^3}\begin{pmatrix} \eta^{r\dagger}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right) & -\eta^{r\dagger}\left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & -2m^2A + m^2\overline{B}_+ - (\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_- \\ 2m^2A + m^2B_+ - (\boldsymbol{p}\cdot\boldsymbol{k})B_- & 0 \end{pmatrix}\begin{pmatrix} \left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\xi^s \\ \left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\xi^s \end{pmatrix} \\
&= i\frac{e^2}{2m^3}\eta^{r\dagger}\left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\left[-2m^2A + m^2\overline{B}_+ - (\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_-\right]\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\xi^s \\
&\quad - i\frac{e^2}{2m^3}\eta^{r\dagger}\left(1 + \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\left[2m^2A + m^2B_+ - (\boldsymbol{p}\cdot\boldsymbol{k})B_-\right]\left(1 - \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{2m}\right)\xi^s \\
&\approx i\frac{e^2}{2m^3}\eta^{r\dagger}\left\{-2m^2A + m^2\overline{B}_+ - (\boldsymbol{p}\cdot\boldsymbol{k})\overline{B}_- + \frac{m}{2}[\overline{B}_+, \boldsymbol{\sigma}\cdot\boldsymbol{p}]\right\}\xi^s \\
&\quad - i\frac{e^2}{2m^3}\eta^{r\dagger}\left\{2m^2A + m^2B_+ - (\boldsymbol{p}\cdot\boldsymbol{k})B_- - \frac{m}{2}[B_+, \boldsymbol{\sigma}\cdot\boldsymbol{p}]\right\}\xi^s \\
&= i\frac{e^2}{2m^3}\eta^{r\dagger}\left\{-4m^2A + m^2(\overline{B}_+ - B_+) + (\boldsymbol{p}\cdot\boldsymbol{k})(B_- - \overline{B}_-) + \frac{m}{2}[B_+ + \overline{B}_+, \boldsymbol{\sigma}\cdot\boldsymbol{p}]\right\}\xi^s \\
&= -i\frac{2e^2}{m}\eta^{r\dagger}\left[(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_1^*)(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{p}\cdot\boldsymbol{\epsilon}_2^*)(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) - i(\boldsymbol{\epsilon}_1^*\times\boldsymbol{\epsilon}_2^*)\cdot\boldsymbol{k} + \frac{1}{m^2}(\boldsymbol{p}\cdot\boldsymbol{k})(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right]\xi^s
\end{aligned}
$$

となる．

ここで，

$$
i\mathcal{M}^{(1)j} = -i\frac{2e^2}{m}\eta^{r\dagger}\left[\epsilon_1^{j*}(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + \epsilon_2^{j*}(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{k^j}{m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right]\xi^s \tag{5.5.19}
$$

とおけば,

$$i\mathcal{M}(e_s^- e_r^+ \to 2\gamma) = i\mathcal{M}^{(0)} + \sum_j p^j i\mathcal{M}^{(1)j} \tag{5.5.20}$$

とかける.

$^3P_0$ ポジトロニウムの構成

$|^3P_0;0\rangle$ は [5.5.8] で $S=1,\ l=1,\ J=0,\ M=0$ なので

$$\begin{aligned}|^3P_0;0\rangle &= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\sum_{m=-1}^{1}\langle 1,1;m,-m|1,1;00\rangle\,\tilde{\psi}_{1,m}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,|1,-m\rangle \\ &= \frac{1}{\sqrt{3m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}_{11}\,|{\downarrow\downarrow}\rangle - \tilde{\psi}_{10}\frac{|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle}{\sqrt{2}} + \tilde{\psi}_{1,-1}\,|{\uparrow\uparrow}\rangle\right]\end{aligned}$$

となる．ここで

$$\tilde{\psi}^1 = \frac{\tilde{\psi}_{1,-1} - \tilde{\psi}_{1,1}}{\sqrt{2}}, \qquad \tilde{\psi}^2 = i\frac{\tilde{\psi}_{1,-1} + \tilde{\psi}_{1,1}}{\sqrt{2}}, \qquad \tilde{\psi}^3 = \tilde{\psi}_{1,0} \tag{5.5.21}$$

とおけば,

$$|^3P_0;0\rangle = \frac{1}{\sqrt{6m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}^1(|{\uparrow\uparrow}\rangle - |{\downarrow\downarrow}\rangle) - i\tilde{\psi}^2(|{\uparrow\uparrow}\rangle + |{\downarrow\downarrow}\rangle) - \tilde{\psi}^3(|{\uparrow\downarrow}\rangle + |{\downarrow\uparrow}\rangle)\right]$$

となる.

2^3P_0 ポジトロニウムの崩壊

$n=2$ の場合を考える.

$$\begin{aligned}\psi^1(\boldsymbol{x}) &= \frac{1}{4\sqrt{2\pi}}\frac{1}{{a_0}^{5/2}}x\exp\left(-\frac{r}{2a_0}\right)\\ \psi^2(\boldsymbol{x}) &= \frac{1}{4\sqrt{2\pi}}\frac{1}{{a_0}^{5/2}}y\exp\left(-\frac{r}{2a_0}\right)\\ \psi^3(\boldsymbol{x}) &= \frac{1}{4\sqrt{2\pi}}\frac{1}{{a_0}^{5/2}}z\exp\left(-\frac{r}{2a_0}\right)\end{aligned}$$

なので,

$$\frac{\partial\psi^i}{\partial x^j}(0) = \frac{\delta^{ij}}{4\sqrt{2\pi}{a_0}^{5/2}}.$$

ポジトロニウムの崩壊の不変振幅は

$$\begin{aligned}i\mathcal{M}(2^3P_0\to 2\gamma) = \frac{1}{\sqrt{6m}}\int\frac{d^3p}{(2\pi)^3}\Big[&\tilde{\psi}^1(\boldsymbol{p})\left\{i\mathcal{M}(e_\uparrow^- e_\uparrow^+ \to 2\gamma) - i\mathcal{M}(e_\downarrow^- e_\downarrow^+ \to 2\gamma)\right\}\\ &- i\tilde{\psi}^2(\boldsymbol{p})\left\{i\mathcal{M}(e_\uparrow^- e_\uparrow^+ \to 2\gamma) + i\mathcal{M}(e_\downarrow^- e_\downarrow^+ \to 2\gamma)\right\}\\ &- \tilde{\psi}^3(\boldsymbol{p})\left\{i\mathcal{M}(e_\uparrow^- e_\downarrow^+ \to 2\gamma) + i\mathcal{M}(e_\downarrow^- e_\uparrow^+ \to 2\gamma)\right\}\Big]\end{aligned}$$

で与えられる．[5.5.20] のように，不変振幅は $i\mathcal{M} = i\mathcal{M}^{(0)} + \sum ip^j\mathcal{M}^{(1)j}$ と表すことができたが,

$$\int\frac{d^3p}{(2\pi)^3}\tilde{\psi}^i i\mathcal{M}^{(0)} = \psi^i(0)i\mathcal{M}^{(0)} = 0$$

である．さらに，

$$\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}^i \sum_j ip^j \mathcal{M}^{(1)j} = i\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}} i\mathcal{M}^{(1)i}$$

なので

$$\int \frac{d^3p}{(2\pi)^3}\tilde{\psi}^i i\mathcal{M} = i\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}} i\mathcal{M}^{(1)i}$$

となる．従って

$$\begin{aligned}
i\mathcal{M}(2^3P_0 \to 2\gamma) = \frac{1}{\sqrt{6m}}\frac{i}{4\sqrt{2\pi}{a_0}^{5/2}}\Big[&\Big\{i\mathcal{M}^{(1)1}(e^-_\uparrow e^+_\uparrow \to 2\gamma) - i\mathcal{M}^{(1)1}(e^-_\downarrow e^+_\downarrow \to 2\gamma)\Big\} \\
&- i\Big\{i\mathcal{M}^{(1)2}(e^-_\uparrow e^+_\uparrow \to 2\gamma) + i\mathcal{M}^{(1)2}(e^-_\downarrow e^+_\downarrow \to 2\gamma)\Big\} \\
&- \Big\{i\mathcal{M}^{(1)3}(e^-_\uparrow e^+_\downarrow \to 2\gamma) + i\mathcal{M}^{(1)3}(e^-_\downarrow e^+_\uparrow \to 2\gamma)\Big\}\Big]
\end{aligned}$$

となる．[5.5.7] から

$$\xi^\uparrow\eta^{\uparrow\dagger} - \xi^\downarrow\eta^{\downarrow\dagger} = \sigma^1, \qquad -i(\xi^\uparrow\eta^{\uparrow\dagger} + \xi^\downarrow\eta^{\downarrow\dagger}) = \sigma^2, \qquad -(\xi^\uparrow\eta^{\downarrow\dagger} + \xi^\downarrow\eta^{\uparrow\dagger}) = \sigma^3$$

である．[5.5.19] は

$$\begin{aligned}
i\mathcal{M}^{(1)j} &= -i\frac{2e^2}{m}\eta^{r\dagger}\left[\epsilon_1^{j*}(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + \epsilon_2^{j*}(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{k^j}{m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right]\xi^s \\
&= -i\frac{2e^2}{m}\,\mathrm{Tr}\left[\xi^s\eta^{r\dagger}\left\{\epsilon_1^{j*}(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + \epsilon_2^{j*}(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{k^j}{m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right\}\right]
\end{aligned}$$

と表せるので，

$$\begin{aligned}
i\mathcal{M}(2^3P_0 \to 2\gamma) = &\frac{1}{\sqrt{6m}}\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}}\frac{2e^2}{m} \\
&\times \mathrm{Tr}\left[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{1}{m^2}(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right].
\end{aligned}$$

$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ から

$$\begin{aligned}
i\mathcal{M}(2^3P_0 \to 2\gamma) &= \frac{e^2}{4\sqrt{3\pi m}m{a_0}^{5/2}}\,\mathrm{Tr}\left[2(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*) + \frac{|\boldsymbol{k}|^2}{m^2}(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right] \\
&= \frac{e^2}{2\sqrt{3\pi m}m{a_0}^{5/2}}\frac{2m^2+|\boldsymbol{k}|^2}{m^2}(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*).
\end{aligned}$$

光子の偏極ベクトルの完全性

$$\sum_{\text{polarization}} \epsilon^{i*}(k)\epsilon^j(k) = \delta^{ij} - \frac{k^ik^j}{|\boldsymbol{k}|^2}$$

から

$$\sum_{\text{polarization}} |\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*|^2 = \sum_{\text{pol 1}}\sum_{\text{pol 2}}\sum_{i=1}^3\sum_{j=1}^3 \epsilon_1^{i*}\epsilon_1^j\epsilon_2^{i*}\epsilon_2^j = \sum_{i=1}^3\sum_{j=1}^3\left(\delta^{ij} - \frac{k^ik^j}{|\boldsymbol{k}|^2}\right)\left(\delta^{ij} - \frac{k^ik^j}{|\boldsymbol{k}|^2}\right) = 2$$

なので

$$\sum_{\text{polarization}} |\mathcal{M}(2^3P_0 \to 2\gamma)|^2 = \frac{e^4}{12\pi m^3{a_0}^5}\left(\frac{2m^2+|\boldsymbol{k}|^2}{m^2}\right)^2 \sum_{\text{polarization}} |\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*|^2$$

$$
\begin{aligned}
&= \frac{e^4}{6\pi m^3 {a_0}^5}\left(\frac{2m^2+|\boldsymbol{k}|^2}{m^2}\right)^2 \\
&= \frac{\pi}{12} m^2\alpha^7 \left(\frac{2m^2+|\boldsymbol{k}|^2}{m^2}\right)^2 .
\end{aligned}
$$

(4.86) から

$$
\begin{aligned}
\Gamma(2^3P_0 \to 2\gamma) &= \frac{1}{2}\frac{1}{4m}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{4E_1E_2}|\mathcal{M}(2^3P_0\to 2\gamma)|^2(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2) \\
&= \frac{1}{8m}\int \frac{d^3k_1}{(2\pi)^2}\frac{1}{4|\boldsymbol{k}|^2}|\mathcal{M}(2^3P_2\to 2\gamma)|^2\,\delta(2E_1-2m) \\
&= \frac{1}{256\pi^2 m}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\,|\mathcal{M}(^3P_2\to 2\gamma)|^2 \\
&= \frac{1}{256\pi^2 m}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\times 4\pi\frac{\pi}{12}m^2\alpha^7\left(\frac{2m^2+|\boldsymbol{k}|^2}{m^2}\right)^2 \\
&= \frac{3}{256}m\alpha^7 .
\end{aligned}
$$

2^3P_2 ポジトロニウムの崩壊

$n=2$ の場合を考える．

$|2^3P_2;0\rangle$ は $S=1$, $l=1$, $J=2$, $M=0$ なので

$$
\begin{aligned}
|2^3P_2;0\rangle &= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\sum_{m=-1}^{1}\langle 1,1;m,-m|1,1;20\rangle\,\tilde{\psi}_{1,m}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,|1,-m\rangle \\
&= \frac{1}{\sqrt{6m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}_{11}\,|{\downarrow\downarrow}\rangle + 2\tilde{\psi}_{10}\frac{|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle}{\sqrt{2}} + \tilde{\psi}_{1,-1}\,|{\uparrow\uparrow}\rangle\right] \\
&= \frac{1}{2\sqrt{3m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}^1(|{\uparrow\uparrow}\rangle-|{\downarrow\downarrow}\rangle) - i\tilde{\psi}^2(|{\uparrow\uparrow}\rangle+|{\downarrow\downarrow}\rangle) + 2\tilde{\psi}^3(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle)\right].
\end{aligned}
$$

$\boldsymbol{\sigma}' = (\sigma^1,\sigma^2,-2\sigma^3)$ とすれば

$$
\mathrm{Tr}\left[(\boldsymbol{\sigma}'\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})\right] = 2a^1b^1+2a^2b^2-4a^3b^3 .
$$

$M=0$ の不変振幅は

$$
\begin{aligned}
i\mathcal{M}(2^3P_2;0\to 2\gamma) &= \frac{1}{2\sqrt{3m}}\frac{i}{4\sqrt{2\pi}{a_0}^{5/2}}\Big[\Big\{i\mathcal{M}^{(1)1}(e^-_\uparrow e^+_\uparrow\to 2\gamma) - i\mathcal{M}^{(1)1}(e^-_\downarrow e^+_\downarrow\to 2\gamma)\Big\} \\
&\qquad - i\Big\{i\mathcal{M}^{(1)2}(e^-_\uparrow e^+_\uparrow\to 2\gamma) + i\mathcal{M}^{(1)2}(e^-_\downarrow e^+_\downarrow\to 2\gamma)\Big\} \\
&\qquad + 2\Big\{i\mathcal{M}^{(1)3}(e^-_\uparrow e^+_\downarrow\to 2\gamma) + i\mathcal{M}^{(1)3}(e^-_\downarrow e^+_\uparrow\to 2\gamma)\Big\}\Big] \\
&= \frac{1}{2\sqrt{3m}}\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}}\frac{2e^2}{m} \\
&\quad\times\mathrm{Tr}\left[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{1}{m^2}(\boldsymbol{\sigma}'\cdot\boldsymbol{k})(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right] \\
&= \frac{e^2}{2\sqrt{\pi m}m{a_0}^{5/2}}\left[2h_0^{ij}\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)}{m^2}h_0^{ij}k^ik^j\right],\quad h_0^{ij} = \frac{1}{\sqrt{6}}\,\mathrm{diag}(1,1,-2).
\end{aligned}
$$

$|2^3P_2;1\rangle$ は $S=1,\ l=1,\ J=2,\ M=1$ なので

$$
\begin{aligned}
|2^3P_2;1\rangle &= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\sum_{m=0}^{1}\langle 1,1;m,1-m|1,1;21\rangle\,\tilde{\psi}_{1,m}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,|1,1-m\rangle \\
&= \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}_{10}\,|{\uparrow\uparrow}\rangle+\tilde{\psi}_{11}\frac{|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle}{\sqrt{2}}\right] \\
&= \frac{1}{2\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3}\left[-\tilde{\psi}^1(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle)-i\tilde{\psi}^2(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle)+2\tilde{\psi}^3\,|{\uparrow\uparrow}\rangle\right].
\end{aligned}
$$

$\boldsymbol{\sigma}'=(\sigma^3,i\sigma^3,2\sigma^+)$ とすれば

$$
\mathrm{Tr}\left[(\boldsymbol{\sigma}'\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})\right]=2a^1b^3+2ia^2b^3+2a^3b^1+2ia^3b^2.
$$

$M=1$ の不変振幅は

$$
\begin{aligned}
i\mathcal{M}(2^3P_2;1\to 2\gamma) &= \frac{1}{2\sqrt{2m}}\frac{i}{4\sqrt{2\pi}{a_0}^{5/2}}\Big[-\left\{i\mathcal{M}^{(1)1}(e^-_\uparrow e^+_\downarrow\to 2\gamma)+i\mathcal{M}^{(1)1}(e^-_\downarrow e^+_\uparrow\to 2\gamma)\right\} \\
&\qquad -i\left\{i\mathcal{M}^{(1)2}(e^-_\uparrow e^+_\downarrow\to 2\gamma)+i\mathcal{M}^{(1)2}(e^-_\downarrow e^+_\uparrow\to 2\gamma)\right\} \\
&\qquad +2\left\{i\mathcal{M}^{(1)3}(e^-_\uparrow e^+_\uparrow\to 2\gamma)\right\}\Big] \\
&= \frac{1}{2\sqrt{2m}}\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}}\frac{2e^2}{m} \\
&\qquad\times\mathrm{Tr}\left[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma})+(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma})+\frac{1}{m^2}(\boldsymbol{\sigma}'\cdot\boldsymbol{k})(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right] \\
&= \frac{e^2}{2\sqrt{\pi m}m{a_0}^{5/2}}\left[2h_1^{ij}\epsilon_1^{i*}\epsilon_2^{j*}+\frac{(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)}{m^2}h_1^{ij}k^ik^j\right],\quad h_1^{ij}=\frac{1}{2}\begin{pmatrix} & & 1\\ & & i\\ 1 & i & \end{pmatrix}.
\end{aligned}
$$

$|2^3P_2;-1\rangle$ は $S=1,\ l=1,\ J=2,\ M=-1$ なので

$$
\begin{aligned}
|2^3P_2;-1\rangle &= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\sum_{m=-1}^{0}\langle 1,1;m,-1-m|1,1;2,-1\rangle\,\tilde{\psi}_{1,m}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,|1,-1-m\rangle \\
&= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\sum_{m=-1}^{0}\langle 1,1;-m,1+m|1,1;2,1\rangle\,\tilde{\psi}_{1,m}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\,|1,-1-m\rangle \\
&= \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}_{1,-1}\frac{|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle}{\sqrt{2}}+\tilde{\psi}_{10}\,|{\downarrow\downarrow}\rangle\right] \\
&= \frac{1}{2\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}^1(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle)-i\tilde{\psi}^2(|{\uparrow\downarrow}\rangle+|{\downarrow\uparrow}\rangle)+2\tilde{\psi}^3\,|{\downarrow\downarrow}\rangle\right].
\end{aligned}
$$

$\boldsymbol{\sigma}'=(-\sigma^3,i\sigma^3,-2\sigma^-)$ とすれば

$$
\mathrm{Tr}\left[(\boldsymbol{\sigma}'\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})\right]=-2a^1b^3+2ia^2b^3-2a^3b^1+2ia^3b^2.
$$

$M=-1$ の不変振幅は

$$
\begin{aligned}
i\mathcal{M}(2^3P_2;1\to 2\gamma) &= \frac{1}{2\sqrt{2m}}\frac{i}{4\sqrt{2\pi}{a_0}^{5/2}}\Big[\left\{i\mathcal{M}^{(1)1}(e^-_\uparrow e^+_\downarrow\to 2\gamma)+i\mathcal{M}^{(1)1}(e^-_\downarrow e^+_\uparrow\to 2\gamma)\right\} \\
&\qquad -i\left\{i\mathcal{M}^{(1)2}(e^-_\uparrow e^+_\downarrow\to 2\gamma)+i\mathcal{M}^{(1)2}(e^-_\downarrow e^+_\uparrow\to 2\gamma)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + 2\left\{i\mathcal{M}^{(1)3}(e^-_\downarrow e^+_\downarrow \to 2\gamma)\right\}\Bigg] \\
&= \frac{1}{2\sqrt{2m}}\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}}\frac{2e^2}{m} \\
&\quad \times \mathrm{Tr}\left[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{1}{m^2}(\boldsymbol{\sigma}'\cdot\boldsymbol{k})(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right] \\
&= \frac{e^2}{2\sqrt{\pi m}m{a_0}^{5/2}}\left[2h^{ij}_{-1}\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)}{m^2}h^{ij}_{-1}k^ik^j\right], \quad h^{ij}_{-1} = \frac{1}{2}\begin{pmatrix} & & -1 \\ & & i \\ -1 & i & \end{pmatrix}.
\end{aligned}
$$

$|2^3P_2;2\rangle$ は $S=1,\ l=1,\ J=2,\ M=2$ なので（$m=S_z=1$）

$$
\begin{aligned}
|2^3P_2;2\rangle &= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\langle 1,1;1,1|1,1;22\rangle\,\tilde{\psi}_{1,1}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}|1,1\rangle \\
&= \frac{1}{\sqrt{m}}\int\frac{d^3p}{(2\pi)^3}\tilde{\psi}_{11}|\uparrow\uparrow\rangle \\
&= \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3}\left[-\tilde{\psi}^1|\uparrow\uparrow\rangle - i\tilde{\psi}^2|\uparrow\uparrow\rangle\right].
\end{aligned}
$$

$\boldsymbol{\sigma}' = (-\sigma^+, -i\sigma^+, 0)$ とすれば

$$
\mathrm{Tr}\left[(\boldsymbol{\sigma}'\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})\right] = -a^1b^2 - ia^1b^2 - ia^2b^1 + a^2b^2.
$$

$M=2$ の不変振幅は

$$
\begin{aligned}
i\mathcal{M}(2^3P_2;2\to 2\gamma) &= \frac{1}{\sqrt{2m}}\frac{i}{4\sqrt{2\pi}{a_0}^{5/2}}\left[-\left\{i\mathcal{M}^{(1)1}(e^-_\uparrow e^+_\uparrow\to 2\gamma)\right\} - i\left\{i\mathcal{M}^{(1)2}(e^-_\uparrow e^+_\uparrow\to 2\gamma)\right\}\right] \\
&= \frac{1}{\sqrt{2m}}\frac{1}{4\sqrt{2\pi}{a_0}^{5/2}}\frac{2e^2}{m} \\
&\quad \times \mathrm{Tr}\left[(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^*\cdot\boldsymbol{\sigma}')(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\sigma}) + \frac{1}{m^2}(\boldsymbol{\sigma}'\cdot\boldsymbol{k})(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)\right] \\
&= \frac{e^2}{2\sqrt{\pi m}m{a_0}^{5/2}}\left[2h^{ij}_2\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^*\cdot\boldsymbol{\epsilon}_2^*)}{m^2}h^{ij}_2k^ik^j\right], \quad h^{ij}_2 = \frac{1}{2}\begin{pmatrix} -1 & -i & \\ -i & 1 & \\ & & \end{pmatrix}.
\end{aligned}
$$

$|2^3P_2;-2\rangle$ は $S=1,\ l=1,\ J=2,\ M=-2$ なので（$m=S_z=-1$）

$$
\begin{aligned}
|2^3P_2;-2\rangle &= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\langle 1,1;-1,-1|1,1;2,-2\rangle\,\tilde{\psi}_{1,-1}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}|1,-1\rangle \\
&= \sqrt{2M}\int\frac{d^3p}{(2\pi)^3}\langle 1,1;1,1|1,1;22\rangle\,\tilde{\psi}_{1,-1}\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}|1,-1\rangle \\
&= \frac{1}{\sqrt{m}}\int\frac{d^3p}{(2\pi)^3}\tilde{\psi}_{1,-1}|\downarrow\downarrow\rangle \\
&= \frac{1}{\sqrt{2m}}\int\frac{d^3p}{(2\pi)^3}\left[\tilde{\psi}^1|\downarrow\downarrow\rangle - i\tilde{\psi}^2|\downarrow\downarrow\rangle\right].
\end{aligned}
$$

$\boldsymbol{\sigma}' = (-\sigma^-, i\sigma^-, 0)$ とすれば

$$
\mathrm{Tr}\left[(\boldsymbol{\sigma}'\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})\right] = -a^1b^2 + ia^1b^2 + ia^2b^1 + a^2b^2.
$$

$M=-2$ の不変振幅は

$$
i\mathcal{M}(2^3P_2;-2\to 2\gamma) = \frac{1}{\sqrt{2m}}\frac{i}{4\sqrt{2\pi}{a_0}^{5/2}}\left[\left\{i\mathcal{M}^{(1)1}(e^-_\downarrow e^+_\downarrow\to 2\gamma)\right\} - i\left\{i\mathcal{M}^{(1)2}(e^-_\downarrow e^+_\downarrow\to 2\gamma)\right\}\right]
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2m}} \frac{1}{4\sqrt{2\pi}{a_0}^{5/2}} \frac{2e^2}{m} \\
&\qquad \times \mathrm{Tr}\left[(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}')(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}')(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) + \frac{1}{m^2}(\boldsymbol{\sigma}' \cdot \boldsymbol{k})(\boldsymbol{\sigma} \cdot \boldsymbol{k})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)\right] \\
&= \frac{e^2}{2\sqrt{\pi m}m{a_0}^{5/2}} \left[2h_{-2}^{ij}\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)}{m^2} h_{-2}^{ij}k^i k^j\right], \quad h_{-2}^{ij} = \frac{1}{2}\begin{pmatrix} -1 & i & \\ i & 1 & \\ & & \end{pmatrix}.
\end{aligned}
$$

以上から,

$$
i\mathcal{M}(2^3P_2; M \to 2\gamma) = \frac{e^2}{2\sqrt{\pi m}m{a_0}^{5/2}} \left[2h_M^{ij}\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)}{m^2} h_M^{ij}k^i k^j\right].
$$

h_M はトレースが 0 の対称行列

$$
h_0 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}, \quad h_{\pm 1} = \frac{1}{2}\begin{pmatrix} & & \pm 1 \\ & & i \\ \pm 1 & i & \end{pmatrix}, \quad h_{\pm 2} = \frac{1}{2}\begin{pmatrix} -1 & \mp i & \\ \mp i & 1 & \\ & & \end{pmatrix}
$$

で与えられ,

$$
\mathrm{Tr}\left(h_M h_{M'}^\dagger\right) = \sum_{ij} h_M^{ij} h_{M'}^{ij*} = \delta_{MM'} \tag{5.5.22}
$$

を満たす.

光子の偏極と全角運動量の射影について和を取って,

$$
\begin{aligned}
&\sum_{\text{polarization}} \sum_M |\mathcal{M}(2^3P_2; M \to 2\gamma)|^2 \\
&= \frac{e^4}{4\pi m^3 {a_0}^5} \sum_{\text{polarization}} \sum_M \left|2h_M^{ij}\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)}{m^2} h_M^{ij}k^i k^j\right|^2 \\
&= \frac{e^4}{4\pi m^3 {a_0}^5} \sum_{\text{polarization}} \sum_M \left[2h_M^{ij}\epsilon_1^{i*}\epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)}{m^2} h_M^{ij}k^i k^j\right]\left[2h_M^{kl*}\epsilon_1^{k}\epsilon_2^{l} + \frac{(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2)}{m^2} h_M^{kl*}k^k k^l\right] \\
&= \frac{e^4}{4\pi m^3 {a_0}^5} \sum_{\text{pol}} \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \\
&\quad \times \left[4\epsilon_1^{i*}\epsilon_1^k\epsilon_2^{j*}\epsilon_2^l + \frac{2k^k k^l}{m^2}(\boldsymbol{\epsilon}_1 \cdot \boldsymbol{\epsilon}_2)\epsilon_1^{i*}\epsilon_2^{j*} + \frac{2k^i k^j}{m^2}(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)\epsilon_1^k\epsilon_2^l + \frac{k^i k^j k^k k^l}{m^4}|\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*|^2\right] \\
&= \frac{e^4}{4\pi m^3 {a_0}^5} \sum_{\text{pol}} \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \\
&\quad \times \left[4\epsilon_1^{i*}\epsilon_1^k\epsilon_2^{j*}\epsilon_2^l + \frac{2k^k k^l}{m^2}\sum_m \epsilon_1^{i*}\epsilon_1^m\epsilon_2^{j*}\epsilon_2^m + \frac{2k^i k^j}{m^2}\sum_m \epsilon_1^{m*}\epsilon_1^k\epsilon_2^{m*}\epsilon_2^l + \frac{k^i k^j k^k k^l}{m^4}|\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*|^2\right].
\end{aligned}
$$

ϵ^μ の完全性から

$$
\begin{aligned}
&|\mathcal{M}(2^3P_2 \to 2\gamma)|^2 \\
&= \frac{e^4}{4\pi m^3 {a_0}^5} 4 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \left(\delta^{ik} - \frac{k^i k^k}{|\boldsymbol{k}|^2}\right)\left(\delta^{jl} - \frac{k^j k^l}{|\boldsymbol{k}|^2}\right) && [5.5.23] \\
&\quad + \frac{e^4}{4\pi m^3 {a_0}^5} 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \frac{k^k k^l}{m^2} \sum_m \left(\delta^{im} - \frac{k^i k^m}{|\boldsymbol{k}|^2}\right)\left(\delta^{jm} - \frac{k^j k^m}{|\boldsymbol{k}|^2}\right) && [5.5.24]
\end{aligned}
$$

$$+\frac{e^4}{4\pi m^3 {a_0}^5}2\sum_M\sum_{ijkl}h_M^{ij}h_M^{kl*}\frac{k^ik^j}{m^2}\sum_m\left(\delta^{mk}-\frac{k^mk^k}{|\boldsymbol{k}|^2}\right)\left(\delta^{ml}-\frac{k^mk^l}{|\boldsymbol{k}|^2}\right) \qquad [5.5.25]$$

$$+\frac{e^4}{4\pi m^3 {a_0}^5}2\sum_M\sum_{ijkl}h_M^{ij}h_M^{kl*}\frac{k^ik^jk^kk^l}{m^4}. \qquad [5.5.26]$$

(4.86) から

$$\begin{aligned}\Gamma(2^3P_2\to2\gamma)&=\frac{1}{2}\frac{1}{4m}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{4E_1E_2}|\mathcal{M}(2^3P_2\to2\gamma)|^2(2\pi)^4\,\delta^{(4)}(k_1+k_2-p_1-p_2)\\&=\frac{1}{8m}\int\frac{d^3k_1}{(2\pi)^2}\frac{1}{4|\boldsymbol{k}|^2}|\mathcal{M}(2^3P_2\to2\gamma)|^2\,\delta(2E_1-2m)\\&=\frac{1}{256\pi^2m}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\,|\mathcal{M}(2^3P_2\to2\gamma)|^2.\end{aligned}$$

$|\mathcal{M}|^2$ の中に現れる積分を計算する．まず，

$$\int d\Omega\,\frac{k^ik^j}{|\boldsymbol{k}|^2}$$

は $i\neq j$ ならば 0 なので，δ^{ij} に比例する：

$$\int d\Omega\,\frac{k^ik^j}{|\boldsymbol{k}|^2}=A\delta^{ij}.$$

$i=j=1,2,3$ について和を取れば $4\pi=3A$ となるので，

$$\int d\Omega\,\frac{k^ik^j}{|\boldsymbol{k}|^2}=\frac{4\pi}{3}\delta^{ij}.$$

次に，

$$\int d\Omega\,\frac{k^ik^jk^kk^l}{|\boldsymbol{k}|^4}=B(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})$$

とおく．$i=j=1,2,3$ について和を取れば

$$\int d\Omega\,\frac{k^kk^l}{|\boldsymbol{k}|^2}=\frac{4\pi}{3}\delta^{kl}=B(3\delta^{kl}+\delta^{kl}+\delta^{kl})=5B\delta^{kl}$$

なので

$$\int d\Omega\,\frac{k^ik^jk^kk^l}{|\boldsymbol{k}|^4}=\frac{4\pi}{15}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}).$$

[5.5.22] に注意して [5.5.23] を積分すれば

$$\begin{aligned}&4\sum_M\sum_{ijkl}h_M^{ij}h_M^{kl*}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\left(\delta^{ik}-\frac{k^ik^k}{|\boldsymbol{k}|^2}\right)\left(\delta^{jl}-\frac{k^jk^l}{|\boldsymbol{k}|^2}\right)\\&=4\sum_M\sum_{ijkl}h_M^{ij}h_M^{kl*}\left[4\pi\delta^{ik}\delta^{jl}-\frac{4\pi}{3}\delta^{ik}\delta^{jl}-\frac{4\pi}{3}\delta^{ik}\delta^{jl}+\frac{4\pi}{15}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})\right]\\&=4\sum_M\sum_{ijkl}h_M^{ij}h_M^{kl*}\left[\frac{4\pi}{15}(\delta^{ij}\delta^{kl}+\delta^{il}\delta^{jk})+\frac{8\pi}{5}\delta^{ik}\delta^{jl}\right]\\&=4\sum_M\sum_{ij}\left[\frac{4\pi}{15}h_M^{ij}h_M^{ji*}+\frac{8\pi}{5}h_M^{ij}h_M^{ij*}\right]\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_M \frac{12\pi}{15}\delta_{MM} \\
&= \frac{48\pi}{5}.
\end{aligned}
$$

[5.5.24][5.5.25] を積分すれば

$$
\begin{aligned}
&2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\,\frac{k^k k^l}{m^2}\sum_m\left(\delta^{im}-\frac{k^i k^m}{|\boldsymbol{k}|^2}\right)\left(\delta^{jm}-\frac{k^j k^m}{|\boldsymbol{k}|^2}\right) \\
&= 2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\,\frac{k^k k^l}{m^2}\left(\delta^{ij}-\frac{k^i k^j}{|\boldsymbol{k}|^2}\right) \\
&= 2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\,\left(\frac{k^k k^l}{|\boldsymbol{k}|^2}\delta^{ij}-\frac{k^i k^j k^k k^l}{|\boldsymbol{k}|^4}\right) \\
&= 2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\left[\frac{4\pi}{3}\delta^{ij}\delta^{kl}-\frac{4\pi}{15}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})\right] \\
&= 2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\left[\frac{16\pi}{15}\delta^{ij}\delta^{kl}-\frac{4\pi}{15}(\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk})\right] \\
&= 2\sum_M\sum_{ij}\left[-\frac{4\pi}{15}h_M^{ij}h_M^{ij*}-\frac{4\pi}{15}h_M^{ij}h_M^{ji*}\right] \\
&= -\frac{16\pi}{15}\sum_M\delta_{MM} \\
&= -\frac{16\pi}{5}.
\end{aligned}
$$

[5.5.26] を積分すれば

$$
\begin{aligned}
&2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\int d|\boldsymbol{k}|\,\delta(|\boldsymbol{k}|-m)\int d\Omega\,\frac{k^i k^j k^k k^l}{m^4} \\
&= 2\sum_M\sum_{ijkl} h_M^{ij}h_M^{kl*}\frac{4\pi}{15}(\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}) \\
&= 2\sum_M\sum_{ijkl}\frac{4\pi}{15}(h_M^{ij}h_M^{ij*}+h_M^{ij}h_M^{ji*}) \\
&= 2\sum_M\sum_{ijkl}\frac{4\pi}{15}(h_M^{ij}h_M^{ij*}+h_M^{ij}h_M^{ji*}) \\
&= \frac{16\pi}{15}\sum_M\delta_{MM} \\
&= \frac{16\pi}{5}.
\end{aligned}
$$

以上から

$$
\Gamma(2^3P_2\to 2\gamma)=\frac{1}{256\pi^2 m}\frac{e^4}{4\pi m^3 {a_0}^5}\left(\frac{48\pi}{5}-\frac{16\pi}{5}-\frac{16\pi}{5}+\frac{16\pi}{5}\right)=\frac{m\alpha^7}{320}.
$$

Problem 5.6

使う式を列挙しておく．Fierz 恒等式：

$$\overline{u}_L(p_1)\gamma^\mu u_L(p_2)[\gamma^\mu]_{ab} = \overline{u}_R(p_2)\gamma^\mu u_R(p_1) = 2[u_L(p_2)\overline{u}_L(p_1) + u_R(p_1)\overline{u}_R(p_2)]_{ab}.$$

s, t の定義と性質：

$$\begin{aligned} &s(p_1,p_2) = \overline{u}_R(p_1)u_L(p_2), && t(p_1,p_2) = \overline{u}_L(p_1)u_R(p_2), && \\ &t(p_1,p_2) = (s(p_2,p_1))^*, && s(p_1,p_2) = s(p_2,p_1), && |s(p_1,p_2)|^2 = 2p_1\cdot p_2. \end{aligned}$$

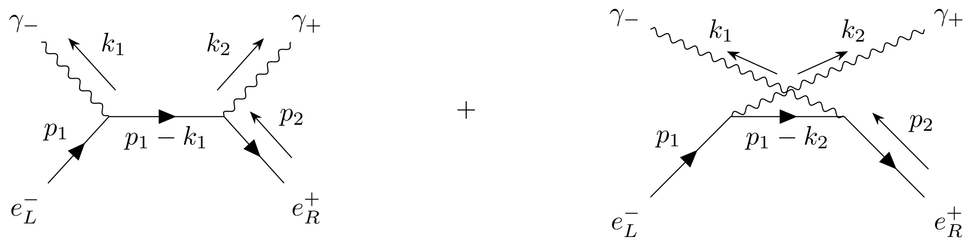
射影に関する性質

$$u_L(p)\overline{u}_L(p) + u_R(p)\overline{u}_R(p) = \not{p}$$

及び[*3]

$$\overline{u}_L u_L = \overline{u}_R u_R = 0.$$

$e_L^- e_R^+ \to \gamma_- \gamma_+$ の過程を考える．



(a) で定義した偏極ベクトルを使い，不変振幅は

$$\begin{aligned} i\mathcal{M} &= -ie^2\epsilon^*_{-\mu}(k_1)\epsilon^*_{+\nu}(k_2)\overline{u}_L(p_2)\left[\gamma^\nu\frac{\not{p}_1 - \not{k}_1}{(p_1-k_1)^2}\gamma^\mu + \gamma^\mu\frac{\not{p}_1 - \not{k}_2}{(p_1-k_2)^2}\gamma^\nu\right]u_L(p_1) \\ &= -ie^2\frac{\overline{u}_L(p_2)\gamma_\mu u_L(k_1)u_R(p_1)\gamma_\nu u_R(k_2)}{4\sqrt{(p_2\cdot k_1)(p_1\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\nu\frac{\not{p}_1 - \not{k}_1}{t}\gamma^\mu + \gamma^\mu\frac{\not{p}_1 - \not{k}_2}{u}\gamma^\nu\right]u_L(p_1) \end{aligned} \qquad [5.5.27]$$

となる[*4]．

[*3] 全ての粒子が質量 0 なので，左巻きスピノルは上半分；右巻きスピノルは下半分の成分のみ持つ．$\overline{u}_L = u_L^\dagger\gamma_0$ は下半分のみ

[*4] $\epsilon(k_1)$ の定義に p_2，$\epsilon(k_2)$ の定義に p_1 を使った

[5.5.27] の第 1 項は，

$$
\begin{aligned}
&\frac{\overline{u}_L(p_2)\gamma_\mu u_L(k_1)u_R(p_1)\gamma_\nu u_R(k_2)}{4\sqrt{(p_2\cdot k_1)(p_1\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{t}\gamma^\mu\right]u_L(p_1)\\
&=-\frac{2}{ut}\overline{u}_L(p_2)\left[u_L(p_1)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(k_1)\overline{u}_L(p_2)+u_R(p_2)\overline{u}_R(k_1)\right]u_L(p_1)\\
&\quad+\frac{2}{ut}\overline{u}_L(p_2)\left[u_L(p_1)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(k_1)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(k_1)\right]\\
&\qquad\qquad\times\left[u_L(k_1)\overline{u}_L(p_2)+u_R(p_2)\overline{u}_R(k_1)\right]u_L(p_1)\\
&=-\frac{2}{ut}\overline{u}_L(p_2)u_R(k_2)\overline{u}_R(p_1)u_L(p_1)\overline{u}_L(p_1)u_R(p_2)\overline{u}_R(k_1)u_L(p_1)\\
&\quad+\frac{2}{ut}\overline{u}_L(p_2)u_R(k_2)\overline{u}_R(p_1)u_L(k_1)\overline{u}_L(k_1)u_R(p_2)\overline{u}_R(k_1)u_L(p_1)\\
&=-\frac{2}{ut}t(p_2,k_2)s(p_1,p_1)t(p_1,p_2)s(k_1,p_1)+\frac{2}{ut}t(p_2,k_2)s(p_1,k_1)t(k_1,p_2)s(k_1,p_1)\\
&=\frac{2}{ut}t(p_2,k_2)s(p_1,k_1)t(k_1,p_2)s(k_1,p_1).
\end{aligned}
\tag{5.5.28}
$$

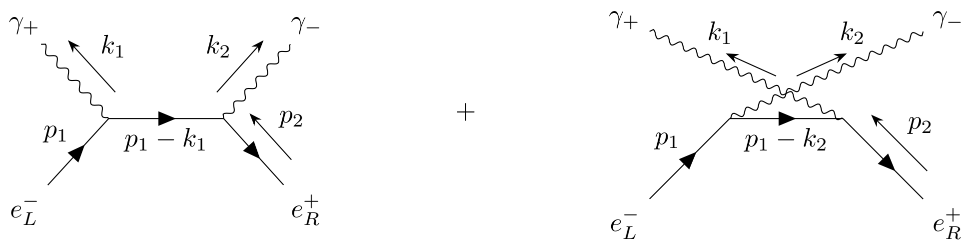
同様に，[5.5.27] の第 2 項は，

$$
\begin{aligned}
&\frac{\overline{u}_L(p_2)\gamma_\mu u_L(k_1)u_R(p_1)\gamma_\nu u_R(k_2)}{4\sqrt{(p_2\cdot k_1)(p_1\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\mu\frac{\not{p}_1-\not{k}_2}{u}\gamma^\nu\right]u_L(p_1)\\
&=-\frac{2}{u^2}\overline{u}_L(p_2)\left[u_L(k_1)\overline{u}_L(p_2)+u_R(p_2)\overline{u}_R(k_1)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_1)\right]u_L(p_1)\\
&\quad+\frac{2}{u^2}\overline{u}_L(p_2)\left[u_L(k_1)\overline{u}_L(p_2)+u_R(p_2)\overline{u}_R(k_1)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(k_2)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_1)\right]u_L(p_1)\\
&=-\frac{2}{u^2}\overline{u}_L(p_2)u_R(p_2)\overline{u}_R(k_1)u_L(p_1)\overline{u}_L(p_1)u_R(k_2)\overline{u}_R(p_1)u_L(p_1)\\
&\quad+\frac{2}{u^2}\overline{u}_L(p_2)u_R(p_2)\overline{u}_R(k_1)u_L(k_2)\overline{u}_L(k_2)u_R(k_2)\overline{u}_R(p_1)u_L(p_1)\\
&=-\frac{2}{u^2}t(p_2,p_2)s(k_1,p_1)t(p_1,k_2)s(p_1,p_1)+\frac{2}{u^2}t(p_2,p_2)s(k_1,k_2)t(k_2,k_2)s(p_1,p_1)\\
&=0.
\end{aligned}
\tag{5.5.29}
$$

[5.5.27][5.5.28][5.5.29] から

$$
i\mathcal{M}(e^-_Le^+_R\to\gamma_-\gamma_+)=-ie^2\frac{2}{ut}t(p_2,k_2)s(p_1,k_1)t(k_1,p_2)s(k_1,p_1).\tag{5.5.30}
$$

$e^-_Le^+_R\to\gamma_+\gamma_-$ の過程を考える．



不変振幅は

$$\begin{aligned} i\mathcal{M} &= -ie^2 \epsilon^*_{+\mu}(k_1)\epsilon^*_{-\nu}(k_2)\overline{u}_L(p_2)\left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{(p_1-k_1)^2}\gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{(p_1-k_2)^2}\gamma^\nu\right]u_L(p_1) \\ &= -ie^2 \frac{\overline{u}_R(p_1)\gamma_\mu u_R(k_1)u_L(p_2)\gamma_\nu u_L(k_2)}{4\sqrt{(p_1\cdot k_1)(p_2\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t}\gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u}\gamma^\nu\right]u_L(p_1) \end{aligned} \tag{5.5.31}$$

となる[*5].

[5.5.31] の第 1 項は,

$$\begin{aligned} &\frac{\overline{u}_R(p_1)\gamma_\mu u_R(k_1)u_L(p_2)\gamma_\nu u_L(k_2)}{4\sqrt{(p_1\cdot k_1)(p_2\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t}\gamma^\mu\right]u_L(p_1) \\ &= -\frac{2}{t^2}\overline{u}_L(p_2)\left[u_L(k_2)\overline{u}_L(p_2) + u_R(p_2)\overline{u}_R(k_2)\right] \\ &\qquad\qquad \times \left[u_L(p_1)\overline{u}_L(p_1) + u_R(p_1)\overline{u}_R(p_1)\right] \\ &\qquad\qquad \times \left[u_L(p_1)\overline{u}_L(k_1) + u_R(k_1)\overline{u}_R(p_1)\right]u_L(p_1) \\ &\quad + \frac{2}{t^2}\overline{u}_L(p_2)\left[u_L(k_2)\overline{u}_L(p_2) + u_R(p_2)\overline{u}_R(k_2)\right] \\ &\qquad\qquad \times \left[u_L(k_1)\overline{u}_L(k_1) + u_R(k_1)\overline{u}_R(k_1)\right] \\ &\qquad\qquad \times \left[u_L(p_1)\overline{u}_L(k_1) + u_R(k_1)\overline{u}_R(p_1)\right]u_L(p_1) \\ &= -\frac{2}{t^2}\overline{u}_L(p_2)u_R(p_2)\overline{u}_R(k_2)u_L(p_1)\overline{u}_L(p_1)u_R(k_1)\overline{u}_R(p_1)u_L(p_1) \\ &\quad + \frac{2}{t^2}\overline{u}_L(p_2)u_R(p_2)\overline{u}_R(k_2)u_L(k_1)\overline{u}_L(k_1)u_R(k_1)\overline{u}_R(p_1)u_L(p_1) \\ &= -\frac{2}{t^2}t(p_2,p_2)s(k_2,p_1)t(p_1,k_1)s(p_1,p_1) + \frac{2}{ut}t(p_2,p_2)s(k_2,k_1)t(k_1,k_1)s(p_1,p_1) \\ &= 0. \end{aligned} \tag{5.5.32}$$

[*5] $\epsilon(k_1)$ の定義に p_1，$\epsilon(k_2)$ の定義に p_2 を使った

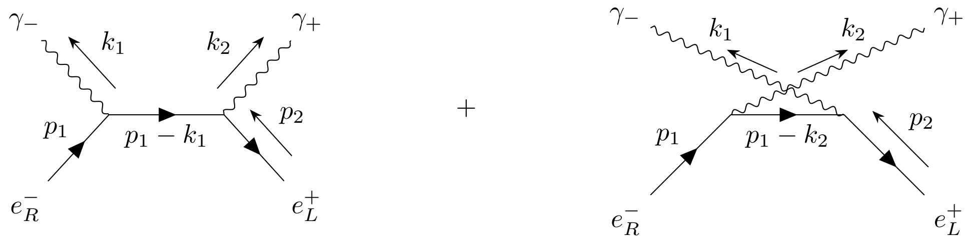
同様に，[5.5.31] の第 2 項は，

$$
\begin{aligned}
&\frac{\overline{u}_R(p_1)\gamma_\mu u_R(k_1)u_L(p_2)\gamma_\nu u_L(k_2)}{4\sqrt{(p_1\cdot k_1)(p_2\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\mu\frac{\not{p}_1-\not{k}_2}{u}\gamma^\nu\right]u_L(p_1)\\
&=-\frac{2}{ut}\overline{u}_L(p_2)\left[u_L(p_1)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(p_2)+u_R(p_2)\overline{u}_R(k_2)\right]u_L(p_1)\\
&\quad+\frac{2}{ut}\overline{u}_L(p_2)\left[u_L(p_1)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(k_2)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(p_2)+u_R(p_2)\overline{u}_R(k_2)\right]u_L(p_1)\\
&=-\frac{2}{ut}\overline{u}_L(p_2)u_R(k_1)\overline{u}_R(p_1)u_L(p_1)\overline{u}_L(p_1)u_R(p_2)\overline{u}_R(k_2)u_L(p_1)\\
&\quad+\frac{2}{ut}\overline{u}_L(p_2)u_R(k_1)\overline{u}_R(p_1)u_L(k_2)\overline{u}_L(k_2)u_R(p_2)\overline{u}_R(k_2)u_L(p_1)\\
&=-\frac{2}{ut}t(p_2,k_1)s(p_1,p_1)t(p_1,p_2)s(k_2,p_1)+\frac{2}{ut}t(p_2,k_1)s(p_1,k_2)t(k_2,p_2)s(k_2,p_1)\\
&=\frac{2}{ut}t(p_2,k_1)s(p_1,k_2)t(k_2,p_2)s(k_2,p_1).
\end{aligned}
\tag{5.5.33}
$$

[5.5.31][5.5.32][5.5.33] から

$$
i\mathcal{M}(e_L^-e_R^+\to\gamma_+\gamma_-)=-ie^2\frac{2}{ut}t(p_2,k_1)s(p_1,k_2)t(k_2,p_2)s(k_2,p_1).
\tag{5.5.34}
$$

$e_R^-e_L^+\to\gamma_-\gamma_+$ の過程を考える．



不変振幅は

$$
\begin{aligned}
i\mathcal{M}&=-ie^2\epsilon^*_{-\mu}(k_1)\epsilon^*_{+\nu}(k_2)\overline{u}_R(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{(p_1-k_1)^2}\gamma^\mu+\gamma^\mu\frac{\not{p}_1-\not{k}_2}{(p_1-k_2)^2}\gamma^\nu\right]u_R(p_1)\\
&=-ie^2\frac{\overline{u}_L(p_1)\gamma_\mu u_L(k_1)u_R(p_2)\gamma_\nu u_R(k_2)}{4\sqrt{(p_1\cdot k_1)(p_2\cdot k_2)}}\overline{u}_R(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{t}\gamma^\mu+\gamma^\mu\frac{\not{p}_1-\not{k}_2}{u}\gamma^\nu\right]u_R(p_1)
\end{aligned}
\tag{5.5.35}
$$

となる[*6]．

[*6] $\epsilon(k_1)$ の定義に p_1，$\epsilon(k_2)$ の定義に p_2 を使った

[5.5.35] の第 1 項は,

$$
\begin{aligned}
&\frac{\overline{u}_L(p_1)\gamma_\mu u_L(k_1)u_R(p_2)\gamma_\nu u_R(k_2)}{4\sqrt{(p_1\cdot k_1)(p_2\cdot k_2)}}\overline{u}_R(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{t}\gamma^\mu\right]u_R(p_1)\\
&=-\frac{2}{t^2}\overline{u}_R(p_2)\left[u_L(p_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_2)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(k_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_1)\right]u_R(p_1)\\
&\quad+\frac{2}{t^2}\overline{u}_R(p_2)\left[u_L(p_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_2)\right]\\
&\qquad\qquad\times\left[u_L(k_1)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(k_1)\right]\\
&\qquad\qquad\times\left[u_L(k_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_1)\right]u_R(p_1)\\
&=-\frac{2}{t^2}\overline{u}_R(p_2)u_L(p_2)\overline{u}_L(k_2)u_R(p_1)\overline{u}_R(p_1)u_L(k_1)\overline{u}_L(p_1)u_R(p_1)\\
&\quad+\frac{2}{t^2}\overline{u}_R(p_2)u_L(p_2)\overline{u}_L(k_2)u_R(k_1)\overline{u}_R(k_1)u_L(k_1)\overline{u}_L(p_1)u_R(p_1)\\
&=-\frac{2}{t^2}s(p_2,p_2)t(k_2,p_1)s(p_1,k_1)t(p_1,p_1)+\frac{2}{ut}s(p_2,p_2)t(k_2,k_1)s(k_1,k_1)t(p_1,p_1)\\
&=0.
\end{aligned}
\tag{5.5.36}
$$

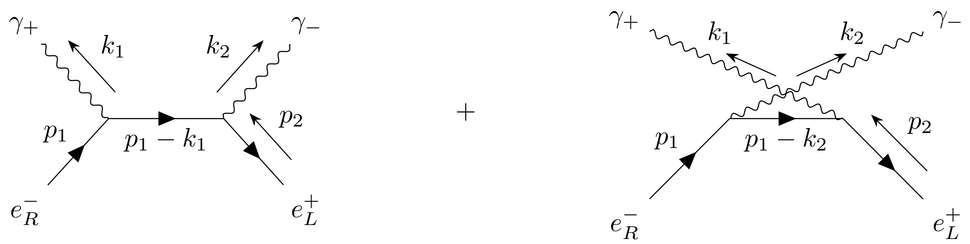
同様に,[5.5.35] の第 2 項は,

$$
\begin{aligned}
&\frac{\overline{u}_L(p_1)\gamma_\mu u_L(k_1)u_R(p_2)\gamma_\nu u_R(k_2)}{4\sqrt{(p_1\cdot k_1)(p_2\cdot k_2)}}\overline{u}_R(p_2)\left[\gamma^\mu\frac{\not{p}_1-\not{k}_2}{u}\gamma^\nu\right]u_R(p_1)\\
&=-\frac{2}{ut}\overline{u}_R(p_2)\left[u_L(k_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_1)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(p_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_2)\right]u_R(p_1)\\
&\quad+\frac{2}{ut}\overline{u}_R(p_2)\left[u_L(k_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_1)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(k_2)\right]\\
&\qquad\qquad\times\left[u_L(p_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(p_2)\right]u_R(p_1)\\
&=-\frac{2}{ut}\overline{u}_R(p_2)u_L(k_1)\overline{u}_L(p_1)u_R(p_1)\overline{u}_R(p_1)u_L(p_2)\overline{u}_L(k_2)u_R(p_1)\\
&\quad+\frac{2}{ut}\overline{u}_R(p_2)u_L(k_1)\overline{u}_L(p_1)u_R(k_2)\overline{u}_R(k_2)u_L(p_2)\overline{u}_L(k_2)u_R(p_1)\\
&=-\frac{2}{ut}s(p_2,k_1)t(p_1,p_1)s(p_1,p_2)t(k_2,p_1)+\frac{2}{ut}s(p_2,k_1)t(p_1,k_2)s(k_2,p_2)t(k_2,p_1)\\
&=\frac{2}{ut}s(p_2,k_1)t(p_1,k_2)s(k_2,p_2)t(k_2,p_1).
\end{aligned}
\tag{5.5.37}
$$

[5.5.35][5.5.36][5.5.37] から

$$
i\mathcal{M}(e_R^-e_L^+\to\gamma_-\gamma_+)=-ie^2\frac{2}{ut}s(p_2,k_1)t(p_1,k_2)s(k_2,p_2)t(k_2,p_1). \tag{5.5.38}
$$

$e_R^-e_L^+\to\gamma_+\gamma_-$ の過程を考える.



(a) で定義した偏極ベクトルを使い，不変振幅は

$$\begin{aligned} i\mathcal{M} &= -ie^2\epsilon^*_{+\mu}(k_1)\epsilon^*_{-\nu}(k_2)\overline{u}_R(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{(p_1-k_1)^2}\gamma^\mu+\gamma^\mu\frac{\not{p}_1-\not{k}_2}{(p_1-k_2)^2}\gamma^\nu\right]u_R(p_1) \\ &= -ie^2\frac{\overline{u}_R(p_2)\gamma_\mu u_R(k_1)u_L(p_1)\gamma_\nu u_L(k_2)}{4\sqrt{(p_2\cdot k_1)(p_1\cdot k_2)}}\overline{u}_R(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{t}\gamma^\mu+\gamma^\mu\frac{\not{p}_1-\not{k}_2}{u}\gamma^\nu\right]u_R(p_1) \end{aligned} \qquad [5.5.39]$$

となる[*7].

[5.5.39] の第 1 項は,

$$\begin{aligned} &\frac{\overline{u}_R(p_2)\gamma_\mu u_R(k_1)u_L(p_1)\gamma_\nu u_L(k_2)}{4\sqrt{(p_2\cdot k_1)(p_1\cdot k_2)}}\overline{u}_L(p_2)\left[\gamma^\nu\frac{\not{p}_1-\not{k}_1}{t}\gamma^\mu\right]u_L(p_1) \\ &= -\frac{2}{ut}\overline{u}_R(p_2)\left[u_L(k_2)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_2)\right] \\ &\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right] \\ &\qquad\times\left[u_L(p_2)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(p_2)\right]u_R(p_1) \\ &\quad+\frac{2}{ut}\overline{u}_R(p_2)\left[u_L(k_2)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_2)\right] \\ &\qquad\times\left[u_L(k_1)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(k_1)\right] \\ &\qquad\times\left[u_L(p_2)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(p_2)\right]u_R(p_1) \\ &= -\frac{2}{ut}\overline{u}_R(p_2)u_L(k_2)\overline{u}_L(p_1)u_R(p_1)\overline{u}_R(p_1)u_L(p_2)\overline{u}_L(k_1)u_R(p_1) \\ &\quad+\frac{2}{ut}\overline{u}_R(p_2)u_L(k_2)\overline{u}_L(p_1)u_R(k_1)\overline{u}_R(k_1)u_L(p_2)\overline{u}_L(k_1)u_R(p_1) \\ &= -\frac{2}{ut}s(p_2,k_2)t(p_1,p_1)s(p_1,p_2)t(k_1,p_1)+\frac{2}{ut}s(p_2,k_2)t(p_1,k_1)s(k_1,p_2)t(k_1,p_1) \\ &= \frac{2}{ut}s(p_2,k_2)t(p_1,k_1)s(k_1,p_2)t(k_1,p_1). \end{aligned} \qquad [5.5.40]$$

[*7] $\epsilon(k_1)$ の定義に p_2，$\epsilon(k_2)$ の定義に p_1 を使った

同様に，[5.5.39] の第 2 項は，

$$
\begin{aligned}
&\frac{\overline{u}_R(p_2)\gamma_\mu u_R(k_1)u_L(p_1)\gamma_\nu u_L(k_2)}{4\sqrt{(p_2\cdot k_1)(p_1\cdot k_2)}}\overline{u}_R(p_2)\left[\gamma^\mu\frac{\not{p}_1-\not{k}_2}{u}\gamma^\nu\right]u_R(p_1)\\
&=-\frac{2}{u^2}\overline{u}_R(p_2)\left[u_L(p_2)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(p_2)\right]\\
&\qquad\qquad\times\left[u_L(p_1)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(p_1)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_2)\right]u_R(p_1)\\
&\quad+\frac{2}{u^2}\overline{u}_R(p_2)\left[u_L(p_2)\overline{u}_L(k_1)+u_R(k_1)\overline{u}_R(p_2)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(k_2)+u_R(k_2)\overline{u}_R(k_2)\right]\\
&\qquad\qquad\times\left[u_L(k_2)\overline{u}_L(p_1)+u_R(p_1)\overline{u}_R(k_2)\right]u_R(p_1)\\
&=-\frac{2}{u^2}\overline{u}_R(p_2)u_L(p_2)\overline{u}_L(k_1)u_R(p_1)\overline{u}_R(p_1)u_L(k_2)\overline{u}_L(p_1)u_R(p_1)\\
&\quad+\frac{2}{u^2}\overline{u}_R(p_2)u_L(p_2)\overline{u}_L(k_1)u_R(k_2)\overline{u}_R(k_2)u_L(k_2)\overline{u}_L(p_1)u_R(p_1)\\
&=-\frac{2}{u^2}s(p_2,p_2)t(k_1,p_1)s(p_1,k_2)t(p_1,p_1)+\frac{2}{u^2}s(p_2,p_2)t(k_1,k_2)s(k_2,k_2)t(p_1,p_1)\\
&=0.
\end{aligned}
\tag{5.5.41}
$$

[5.5.39][5.5.40][5.5.41] から

$$
i\mathcal{M}(e_R^-e_L^+\to\gamma_+\gamma_-)=-ie^2\frac{2}{ut}s(p_2,k_2)t(p_1,k_1)s(k_1,p_2)t(k_1,p_1). \tag{5.5.42}
$$

[5.5.30][5.5.34][5.5.38][5.5.42] から，

$$
\begin{aligned}
|\mathcal{M}(e_L^-e_R^+\to\gamma_-\gamma_+)|^2=|\mathcal{M}(e_R^-e_L^+\to\gamma_+\gamma_-)|^2&=e^4\frac{64}{u^2t^2}(p_2\cdot k_2)(p_1\cdot k_1)(k_1\cdot p_2)(k_1\cdot p_1)\\
&=4e^4\frac{t}{u},\\
|\mathcal{M}(e_L^-e_R^+\to\gamma_+\gamma_-)|^2=|\mathcal{M}(e_R^-e_L^+\to\gamma_-\gamma_+)|^2&=e^4\frac{64}{u^2t^2}(p_2\cdot k_1)(p_1\cdot k_2)(k_2\cdot p_2)(k_2\cdot p_1)\\
&=4e^4\frac{u}{t}.
\end{aligned}
$$

従って，

$$
\frac{1}{4}\sum_{\text{spin}}\sum_{\text{polarization}}|\mathcal{M}|^2=2e^4\left(\frac{u}{t}+\frac{t}{u}\right).
$$

慣性質量から観測すれば，(4.85) から

$$
\frac{d\sigma}{d\Omega}=\frac{|\mathcal{M}|^2}{64\pi^2{E_{\mathrm{cm}}}^2}=\frac{\alpha^2}{2s}\left(\frac{u}{t}+\frac{t}{u}\right)=\frac{\alpha^2}{s}\frac{1+\cos^2\theta}{\sin^2\theta}.
$$

従って，

$$
\frac{d\sigma}{d\cos\theta}=\frac{2\pi\alpha^2}{s}\frac{1+\cos^2\theta}{\sin^2\theta}.
$$

Chapter 6

Radiative Corrections: Introduction

6.3 The Electron Vertex Function: Evaluation

(6.44)

電子の運動量 p, $p' = p + q$ について,

$$p'^2 = (p+q)^2 = m^2$$

が成立する．直ちに

$$2p \cdot q + q^2 = 0$$

が分かる．さらに,

$$x + y + z = 1.$$

よって,

$$\begin{aligned}
\ell^2 - \Delta &= (k + yq - zp)^2 + xyq^2 - (1-z)^2 m^2 \\
&= k^2 + y^2 q^2 + z^2 p^2 + 2yk \cdot q - 2yzp \cdot q - 2zk \cdot p + xyq^2 - (1-z)^2 m^2 \\
&= k^2 + 2k \cdot (yq - zp) + y^2 q^2 + z^2 p^2 - 2yzp \cdot q + xyq^2 - (1-z)^2 m^2 \\
&= k^2 + 2k \cdot (yq - zp) + y(x+y)q^2 - 2yzp \cdot q + \{z^2 - (1-z)^2\} m^2 \\
&= k^2 + 2k \cdot (yq - zp) + y(x+y)q^2 + yzq^2 + (2z-1)m^2 \\
&= k^2 + 2k \cdot (yq - zp) + y(x+y+z)q^2 + (z - x - y)m^2 \\
&= k^2 + 2k \cdot (yq - zp) + yq^2 + zm^2 - (x+y)m^2 \\
&= k^2 + 2k \cdot (yq - zp) + yq^2 + zp^2 - (x+y)m^2 \\
&= D - i\epsilon.
\end{aligned}$$

p.191 **中盤の式**

(6.38) の積分変数 k を $l = k + yq - zp$ に変更する．(6.38) は

$$2ie^2 \int_0^1 dxdydz\, \delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{\overline{u}(p') \left[\not{k} \gamma^\mu \not{k}' + m^2 \gamma^\mu - 2m(k+k')^\mu \right] u(p)}{D^3} \qquad [6.3.1]$$

となる．（積分値が変わらない範囲で；$\to$ と表記する）分子を式変形する．

ここで

$$k = \ell - yq + zp =: \ell + a, \quad k' = k + q =: \ell + b$$

とおく．

$$\not{k}\gamma^\mu \not{k}' = (\not{\ell} + \not{a})\gamma^\mu(\not{\ell} + \not{b}) = \not{\ell}\gamma^\mu\not{\ell} + \not{\ell}\gamma^\mu\not{b} + \not{a}\gamma^\mu\not{\ell} + \not{a}\gamma^\mu\not{b}$$

となるが，$\not{\ell}\gamma^\mu\not{b} = \ell_\nu\gamma^\nu\gamma^\mu\not{b}$ なので，(6.45) から上式の第 2, 3 項の積分は 0 である．従って，

$$\begin{aligned}
&\to \not{\ell}\gamma^\mu\not{\ell} + \not{a}\gamma^\mu\not{b} \\
&= \ell^\nu\gamma^\nu\gamma^\mu\not{\ell} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\
&= \ell^\nu(2g^{\mu\nu} - \gamma^\mu\gamma^\nu)\not{\ell} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\
&= 2\ell^\mu\not{\ell} - \gamma^\mu\not{\ell}\not{\ell} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\
&= 2\ell^\mu\ell^\nu\gamma_\nu - \gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}).
\end{aligned}$$

(6.46) から

$$\begin{aligned}
&\to 2\frac{1}{4}g^{\mu\nu}\ell^2\gamma_\nu - \gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\
&= \frac{1}{2}\gamma^\mu\ell^2 - \gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\
&= -\frac{1}{2}\gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}).
\end{aligned}$$

同様に，

$$-2m(k+k')^\mu = -2m(2l^\mu + a^\mu + b^\mu) \to -2m(a^\mu + b^\mu) = -2m((1-2y)q^\mu + 2zp^\mu).$$

p.192 序盤の式

Dirac 方程式

$$\not{p}u(p) = mu(p), \quad \overline{u}(p')\not{p}' = \overline{u}(p')m, \quad \overline{u}(p')\not{q}u(p) = 0 \tag{6.3.2}$$

に注意する．

$$\overline{u}(p')\left[-\frac{\gamma^\mu}{2}\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\right]u(p)$$

を計算する（これ以降は両端のスピノルを省略する）．2 項目は

$$\begin{aligned}
&(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] \\
&= [-(y+z)\not{q} + z\not{p}']\gamma^\mu[(1-y)\not{q} + z\not{p}] \\
&= [-(y+z)\not{q} + zm]\gamma^\mu[(1-y)\not{q} + zm] \\
&= -(y+z)(1-y)\not{q}\gamma^\mu\not{q} - z(y+z)m\not{q}\gamma^\mu + z(1-y)m\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
&= -(y+z)(1-y)(2q^\mu - \gamma^\mu\not{q})\not{q} \\
&\quad - z(y+z)m(2q^\mu - \gamma^\mu\not{q}) + z(1-y)m\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
&= -2(y+z)(1-y)q^\mu\not{q} + (y+z)(1-y)\gamma^\mu\not{q}\not{q}
\end{aligned}$$

$$
\begin{aligned}
&-2z(y+z)mq^\mu + z(y+z)m\gamma^\mu \not{q} + z(1-y)m\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= -2(y+z)(1-y)q^\mu \not{q} + (y+z)(1-y)\gamma^\mu q^2 \\
&\quad -2z(y+z)mq^\mu + z(1+z)m\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu .
\end{aligned}
$$

$\overline{u}(p')\not{q}u(p) = 0$ なので,

$$
= (y+z)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + z(1+z)m\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu .
$$

$\gamma^\mu \not{q} = \gamma^\mu(\not{p}' - \not{p}) = \gamma^\mu(\not{p}' - m) = (\gamma^\mu \not{p}' - m\gamma^\mu) = (2p'^\mu - \not{p}'\gamma^\mu - m\gamma^\mu) = 2p'^\mu - 2m\gamma^\mu$ なので,

$$
\begin{aligned}
&= (y+z)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + z(1+z)m(2p'^\mu - 2m\gamma^\mu) + z^2 m^2 \gamma^\mu \\
&= (y+z)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu - 2z(1+z)m^2\gamma^\mu + z^2 m^2 \gamma^\mu \\
&= (1-x)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu + (-2z - z^2)m^2 \gamma^\mu .
\end{aligned}
\tag{6.3.3}
$$

他の項も併せれば

$$
\begin{aligned}
&-\frac{\gamma^\mu}{2}\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu) \\
&= -\frac{\gamma^\mu}{2}\ell^2 \\
&\quad + (1-x)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu + (-2z - z^2)m^2\gamma^\mu \\
&\quad + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu) \\
&= -\frac{\gamma^\mu}{2}\ell^2 + (1-x)(1-y)\gamma^\mu q^2 + (1 - 2z - z^2)m^2\gamma^\mu \\
&\quad - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu - 2(1-2y)mq^\mu - 4zmp^\mu .
\end{aligned}
\tag{6.3.4}
$$

2 行目の部分を計算する：

$$
\begin{aligned}
&-2z(y+z)mq^\mu + 2z(1+z)mp'^\mu - 2(1-2y)mq^\mu - 4zmp^\mu \\
&= -4zmp^\mu + 2z(1+z)mp'^\mu - 2z(y+z)mq^\mu - 2(1-2y)mq^\mu \\
&= -4zmp^\mu + 2z(1+z)m(p^\mu + q^\mu) - 2z(y+z)mq^\mu - 2(1-2y)mq^\mu \\
&= 2z(z-1)mp^\mu + 2z(1+z)mq^\mu - 2z(y+z)mq^\mu - 2(1-2y)mq^\mu \\
&= 2z(z-1)mp^\mu + 2z(1-y)mq^\mu - 2(1-2y)mq^\mu \\
&= 2z(z-1)mp^\mu + z(z-1)mq^\mu - z(z-1)mq^\mu + 2z(1-y)mq^\mu - 2(1-2y)mq^\mu \\
&= z(z-1)m(2p^\mu + q^\mu) - z(z-1)mq^\mu + 2z(1-y)mq^\mu - 2(1-2y)mq^\mu \\
&= (p'^\mu + p^\mu)\cdot mz(z-1) + q^\mu \cdot m(-z^2 + z + 2z - 2yz - 2 + 4y) \\
&= (p'^\mu + p^\mu)\cdot mz(z-1) + q^\mu \cdot m(-z^2 + 3z - 2yz - 2 + 4y).
\end{aligned}
\tag{6.3.5}
$$

最後の部分を計算する：

$$
\begin{aligned}
-z^2 + 3z - 2yz - 2 + 4y &= -z(1-x-y) + 3z - 2yz - 2 + 4y \\
&= xz - yz + 4y + 2z - 2 \\
&= xz - yz + 4y + 2(1-x-y) - 2 \\
&= xz - yz - 2x + 2y \\
&= (z-2)(x-y).
\end{aligned}
\tag{6.3.6}
$$

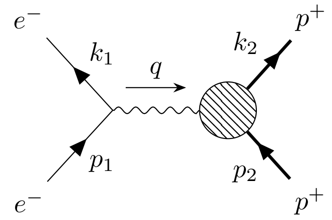
[6.3.4][6.3.5][6.3.6] から

$$
-\frac{\gamma^\mu}{2}\ell^2 + (1-x)(1-y)\gamma^\mu q^2 + (1-2z-z^2)m^2\gamma^\mu + (p'^\mu + p^\mu)\cdot mz(z-1) + q^\mu \cdot m(z-2)(x-y).
$$

Problems

Problem 6.1: Rosenbluth formula

電子と陽子の散乱を考える．



電子の質量は 0，陽子の質量は M とする．スピンの平均を取った不変振幅は

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\operatorname{Tr}[\gamma_\mu \not{p}_1 \gamma_\nu \not{k}_1] \\ &\times \operatorname{Tr}\left[\left(\gamma^\mu(F_1+F_2) - \frac{(p_2+k_2)^\mu}{2M}F_2\right)(\not{p}_2+M)\left(\gamma^\nu(F_1+F_2) - \frac{(p_2+k_2)^\nu}{2M}F_2\right)(\not{k}_2+M)\right]\end{aligned}$$

である．1 つめのトレースは次のように計算できる：

$$\operatorname{Tr}[\gamma_\mu \not{p}_1 \gamma_\nu \not{k}_1] = 4[p_{1\mu}k_{1\nu} + k_{1\mu}p_{1\nu} - g_{\mu\nu}(p_1\cdot k_1)].$$

2 つめのトレースで非零なのはガンマ行列が偶数個含まれる項：

$$\begin{aligned}\operatorname{Tr}[\cdots] &= (F_1+F_2)^2\operatorname{Tr}(\gamma^\mu \not{p}_2 \gamma^\nu \not{k}_2) + {F_2}^2(p_2+k_2)^\mu(p_2+k_2)^\nu \\ &\quad + M^2(F_1+F_2)^2\operatorname{Tr}(\gamma^\mu\gamma^\nu) + \frac{{F_2}^2}{4M^2}(p_2+k_2)^\mu(p_2+k_2)^\nu\operatorname{Tr}(\not{p}_2\not{k}_2) \\ &\quad - \frac{1}{2}F_2(F_1+F_2)(p_2+k_2)^\nu\operatorname{Tr}(\gamma^\mu\not{p}_2) - \frac{1}{2}F_2(F_1+F_2)(p_2+k_2)^\nu\operatorname{Tr}(\gamma^\mu\not{k}_2) \\ &\quad - \frac{1}{2}F_2(F_1+F_2)(p_2+k_2)^\mu\operatorname{Tr}(\gamma^\nu\not{p}_2) - \frac{1}{2}F_2(F_1+F_2)(p_2+k_2)^\mu\operatorname{Tr}(\gamma^\nu\not{k}_2) \\ &= 4(F_1+F_2)^2[p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - g^{\mu\nu}(p_2\cdot k_2)] + {F_2}^2(p_2+k_2)^\mu(p_2+k_2)^\nu \\ &\quad + 4M^2(F_1+F_2)^2 g^{\mu\nu} + \frac{{F_2}^2}{M^2}(p_2\cdot k_2)(p_2+k_2)^\mu(p_2+k_2)^\nu \\ &\quad - 4F_2(F_1+F_2)(p_2+k_2)^\mu(p_2+k_2)^\nu \\ &= 4(F_1+F_2)^2[p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - g^{\mu\nu}(p_2\cdot k_2)] \\ &\quad + 4M^2(F_1+F_2)^2 g^{\mu\nu} \\ &\quad + \left[{F_2}^2 + \frac{{F_2}^2}{M^2}(p_2\cdot k_2) - 4F_2(F_1+F_2)\right](p_2+k_2)^\mu(p_2+k_2)^\nu.\end{aligned}$$

これらの積を求める．1 項目は

$$\begin{aligned}&[p_{1\mu}k_{1\nu} + k_{1\mu}p_{1\nu} - g_{\mu\nu}(p_1\cdot k_1)][p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - g^{\mu\nu}(p_2\cdot k_2)] \\ &= [p_{1\mu}k_{1\nu} + k_{1\mu}p_{1\nu}][p_2^\mu k_2^\nu + k_2^\mu p_2^\nu] \\ &= 2(p_1\cdot p_2)(k_1\cdot k_2) + 2(p_1\cdot k_2)(k_1\cdot p_2).\end{aligned}$$

2 項目は

$$[p_{1\mu}k_{1\nu}+k_{1\mu}p_{1\nu}-g_{\mu\nu}(p_1\cdot k_1)]g^{\mu\nu}=-2(p_1\cdot k_1).$$

3 項目は

$$\begin{aligned}&[p_{1\mu}k_{1\nu}+k_{1\mu}p_{1\nu}-g_{\mu\nu}(p_1\cdot k_1)](p_2+k_2)^\mu(p_2+k_2)^\nu\\&=2p_1\cdot(p_2+k_2)k_1\cdot(p_2+k_2)-(p_1\cdot k_1)(p_2+k_2)^2\\&=2p_1\cdot(p_2+k_2)k_1\cdot(p_2+k_2)-2M^2(p_1\cdot k_1)-2(p_1\cdot k_1)(p_2\cdot k_2).\end{aligned}$$

以上から，

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2&=\frac{8e^4}{q^4}(F_1+F_2)^2[(p_1\cdot p_2)(k_1\cdot k_2)+(p_1\cdot k_2)(k_1\cdot p_2)-M^2(p_1\cdot k_1)]\\&\quad+\frac{2e^4}{q^4}\left[{F_2}^2+\frac{{F_2}^2}{M^2}(p_2\cdot k_2)-4F_2(F_1+F_2)\right]\\&\qquad\times[p_1\cdot(p_2+k_2)k_1\cdot(p_2+k_2)-M^2(p_1\cdot k_1)-(p_1\cdot k_1)(p_2\cdot k_2)].\end{aligned}\tag{6.3.7}$$

ここで，初めに陽子が静止している実験系から見る：

$$p_1^\mu=(E,0,0,E),\qquad k_1^\mu=(E',E'\sin\theta,0,E'\cos\theta),\qquad p_2^\mu=(M,0,0,0).$$

$p_1+p_2=k_1+k_2$ から

$$k_2^\mu=(M+E-E',-E'\sin\theta,0,E-E'\cos\theta),\qquad E'=\frac{ME}{M+2E\sin^2\theta/2}.$$

$q=p_1-k_1$ なので，

$$q^2={p_1}^2+{k_1}^2-2p_1\cdot k_1=-2p_1\cdot k_1.$$

従って，

$$\begin{aligned}(p_1\cdot p_2)(k_1\cdot k_2)&=ME(ME'+EE'-EE'\cos\theta)\\&=MEE'\left(M+2E\sin^2\theta/2\right)\\&=M^2E^2,\\(p_1\cdot k_2)(k_1\cdot p_2)&=(ME-EE'+EE'\cos\theta)ME'\\&=ME'ME'\\&=M^2E'^2,\\p_1\cdot k_1&=-\frac{1}{2}q^2.\end{aligned}$$

以上から，[6.3.7] の第 1 項は

$$\begin{aligned}&\frac{8e^4}{q^4}(F_1+F_2)^2[(p_1\cdot p_2)(k_1\cdot k_2)+(p_1\cdot k_2)(k_1\cdot p_2)-M^2(p_1\cdot k_1)]\\&=\frac{4e^4M^2}{q^4}(F_1+F_2)^2[2E^2+2E'^2+q^2].\end{aligned}$$

$q=k_2-p_2$ なので，

$$k_2\cdot p_2=M^2-\frac{q^2}{2}.$$

従って，

$$
\begin{aligned}
&{F_2}^2 + \frac{{F_2}^2}{M^2}(p_2 \cdot k_2) - 4F_2(F_1 + F_2) \\
&= {F_2}^2 + \frac{{F_2}^2}{M^2}\left(M^2 - \frac{q^2}{2}\right) - 4F_2(F_1 + F_2) \\
&= -2\left(2F_1F_2 + {F_2}^2 + \frac{{F_2}^2 q^2}{4M^2}\right)
\end{aligned}
$$

及び

$$
\begin{aligned}
&p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - M^2(p_1 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_2) \\
&= M(E + E')M(E + E') + M^2\frac{q^2}{2} + \left(M^2 - \frac{q^2}{2}\right)\frac{q^2}{2} \\
&= M^2\left[(E + E')^2 + q^2 - \frac{q^4}{4M^2}\right]
\end{aligned}
$$

を得る．これらから，[6.3.7] の第 2 項は

$$
\begin{aligned}
&\frac{2e^4}{q^4}\left[{F_2}^2 + \frac{{F_2}^2}{M^2}(p_2 \cdot k_2) - 4F_2(F_1 + F_2)\right] \\
&\qquad \times [p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - M^2(p_1 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_2)] \\
&= -\frac{4e^4M^2}{q^4}\left[(E + E')^2 + q^2 - \frac{q^4}{4M^2}\right]\left(2F_1F_2 + {F_2}^2 + \frac{{F_2}^2 q^2}{4M^2}\right) \\
&= -\frac{4e^4M^2}{q^4}\left[(E + E')^2 + q^2 - \frac{q^4}{4M^2}\right]\left((F_1 + F_2)^2 - {F_1}^2 + \frac{{F_2}^2 q^2}{4M^2}\right).
\end{aligned}
$$

以上から，

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{4e^4M^2}{q^4}(F_1 + F_2)^2[2E^2 + 2E'^2 + q^2] \\
&\quad - \frac{4e^4M^2}{q^4}\left[(E + E')^2 + q^2 - \frac{q^4}{4M^2}\right]\left((F_1 + F_2)^2 - {F_1}^2 + \frac{{F_2}^2 q^2}{4M^2}\right) \\
&= \frac{4e^4M^2}{q^4}(F_1 + F_2)^2\left[2E^2 + 2E'^2 - (E + E')^2 + \frac{q^4}{4M^2}\right] \\
&\quad - \frac{4e^4M^2}{q^4}\left[(E + E')^2 + q^2 - \frac{q^4}{4M^2}\right]\left(-{F_1}^2 + \frac{{F_2}^2 q^2}{4M^2}\right) \\
&= \frac{4e^4M^2}{q^4}(F_1 + F_2)^2\left[(E - E')^2 + \frac{q^4}{4M^2}\right] \\
&\quad + \frac{4e^4M^2}{q^4}\left[(E + E')^2 + q^2 - \frac{q^4}{4M^2}\right]\left({F_1}^2 - \frac{{F_2}^2 q^2}{4M^2}\right).
\end{aligned}
$$

ここで，$p_2 \cdot k_2 = M^2 + M(E - E') = M^2 - q^2/2$ なので，

$$
E - E' = -\frac{q^2}{2M}.
$$

$p_1 \cdot k_1 = -q^2/2 = EE'(1 - \cos\theta) = 2EE'\sin^2\theta/2$ なので，

$$
q^2 = -4EE'\sin^2\frac{\theta}{2}.
$$

これらを使えば,

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{4e^4M^2}{q^4}(F_1+F_2)^2\frac{q^4}{2M^2} \\
&\quad + \frac{4e^4M^2}{q^4}\left[E^2+E'^2+2EE'-4EE'\sin^2\frac{\theta}{2}-\frac{q^4}{4M^2}\right]\left(F_1{}^2-\frac{F_2{}^2q^2}{4M^2}\right) \\
&= \frac{4e^4M^2}{q^4}(F_1+F_2)^2\frac{q^4}{2M^2} \\
&\quad + \frac{4e^4M^2}{q^4}\left[(E-E')^2+4EE'\cos^2\frac{\theta}{2}-\frac{q^4}{4M^2}\right]\left(F_1{}^2-\frac{F_2{}^2q^2}{4M^2}\right) \\
&= \frac{4e^4M^2}{q^4}(F_1+F_2)^2\frac{q^4}{2M^2}+\frac{4e^4M^2}{q^4}\left(F_1{}^2-\frac{F_2{}^2q^2}{4M^2}\right)4EE'\cos^2\frac{\theta}{2} \\
&= \frac{4e^4M^2}{q^4}(F_1+F_2)^2\frac{q^2}{2M^2}\left(-4EE'\sin^2\frac{\theta}{2}\right)+\frac{4e^4M^2}{q^4}\left(F_1{}^2-\frac{F_2{}^2q^2}{4M^2}\right)4EE'\cos^2\frac{\theta}{2} \\
&= \frac{16e^4M^2}{q^4}EE'\left[\left(F_1{}^2-\frac{q^2}{4M^2}F_2{}^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right] \\
&= \frac{16e^4M^3E^2}{q^4(M+2E\sin^2\theta/2)}\left[\left(F_1{}^2-\frac{q^2}{4M^2}F_2{}^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right] \\
&= \frac{e^4M(M+2E\sin^2\theta/2)}{E^2\sin^4\theta/2}\left[\left(F_1{}^2-\frac{q^2}{4M^2}F_2{}^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right] \\
&= \frac{16\pi^2\alpha^2M(M+2E\sin^2\theta/2)}{E^2\sin^4\theta/2}\left[\left(F_1{}^2-\frac{q^2}{4M^2}F_2{}^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right].
\end{aligned}
$$

(A.56) から

$$
\begin{aligned}
d\sigma &= \frac{1}{4EM}\frac{d^3k_1d^3k_2}{(2\pi)^6 4E_1E_2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2(2\pi)^4\,\delta^{(4)}(p_1+p_2-k_1-k_2) \\
&= \frac{1}{4EM}\frac{d^3k_1}{(2\pi)^2 4E'E_2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\,\delta(E+M-E'-E_2) \\
&= \frac{1}{4EM}\frac{E'd\cos\theta dE'}{(2\pi)4E_2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\,\delta(E+M-E'-E_2).
\end{aligned}
$$

$\boldsymbol{k}_2=\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{k}_1$ なので,

$$
|\boldsymbol{k}_2|^2 = E^2+E'^2-2EE'\cos\theta.
$$

さらに,

$$
E_2{}^2 = |\boldsymbol{k}_2|^2+M^2 = M^2+E^2+E'^2-2EE'\cos\theta.
$$

よって,

$$
\begin{aligned}
\frac{d\sigma}{d\cos\theta} &= \frac{1}{4EM}\int\frac{E'dE'}{(2\pi)4E_2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\,\delta(E+M-E'-E_2) \\
&= \frac{1}{4EM}\frac{E'}{(2\pi)4E_2}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\frac{E_2}{E_2+E'-E\cos\theta} \\
&= \frac{1}{32\pi EM}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\frac{E'}{E_2+E'-E\cos\theta}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{32\pi EM}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\frac{E'}{E+M-E\cos\theta} \\
&= \frac{1}{32\pi EM}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\frac{E'}{M+2E\sin^2\theta/2} \\
&= \frac{1}{32\pi}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\frac{1}{(M+2E\sin^2\theta/2)^2} \\
&= \frac{1}{32\pi}\frac{16\pi^2\alpha^2 M}{E^2\sin^4\theta/2(M+2E\sin^2\theta/2)}\left[\left({F_1}^2-\frac{q^2}{4M^2}{F_2}^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right] \\
&= \frac{\pi\alpha^2 M}{2E^2\sin^4\theta/2(M+2E\sin^2\theta/2)}\left[\left({F_1}^2-\frac{q^2}{4M^2}{F_2}^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\frac{\theta}{2}\right].
\end{aligned}
$$

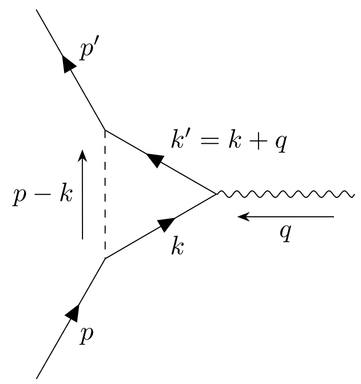
Problem 6.3: Exotic contributuons to $g-2$

(a)

相互作用のハミルトニアンは

$$\mathcal{H}_{\text{int}} = \frac{\lambda}{\sqrt{2}}h\overline{\psi}\psi$$

なので，これは結合定数 $g=\lambda/\sqrt{2}$ の Yukawa 理論 (p.118).



p.189 以降と同様に頂点補正を求める．上図に対応する振幅は

$$\overline{u}(p')\delta\Gamma^\mu(p,p')u(p) = \frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\left[\not{k}'\gamma^\mu\not{k}+m^2\gamma^\mu+m(\not{k}'\gamma^\mu+\gamma^\mu\not{k})\right]u(p)}{[(k-p)^2-{m_h}^2](k'^2-m^2)(k^2-m^2)}$$

で与えられる（以降は両端のスピノル $\overline{u}(p')\cdots u(p)$ は省略する）．

まず，分母を求める．

$$\ell = k+yq-zp, \qquad D=\ell^2-\Delta+i\epsilon, \qquad \Delta = -xyq^2+(1-z)^2m^2+z{m_h}^2$$

とおけば，分母は

$$\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{(\ell^2-\Delta+i\epsilon)^3}.$$

次に分子を求める．

$$\begin{aligned}
\not{k}'\gamma^\mu + \gamma^\mu\not{k} &= \not{q}\gamma^\mu + (\not{k}\gamma^\mu + \gamma^\mu\not{k}) = 2k^\mu + \not{q}\gamma^\mu = 2k^\mu + (\not{p}' - \not{p})\gamma^\mu = 2k^\mu + (m - \not{p})\gamma^\mu \\
&= 2k^\mu + \gamma^\mu(m + \not{p}) - 2p^\mu \\
&= 2k^\mu + 2m\gamma^\mu - 2p^\mu
\end{aligned} \tag{6.3.8}$$

となるので，[6.3.3] と同様の計算を行えば

$$\begin{aligned}
&\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu + m(\not{k}'\gamma^\mu + \gamma^\mu\not{k}) \\
&= -\frac{1}{2}\gamma^\mu\ell^2 + [(1-y)\not{q} + z\not{p}]\gamma^\mu(-y\not{q} + z\not{p}) + m^2\gamma^\mu + 2mk^\mu + 2m^2\gamma^\mu - 2mp^\mu
\end{aligned}$$

となる．第 2 項は

$$\begin{aligned}
&[(1-y)\not{q} + z\not{p}]\gamma^\mu(-y\not{q} + z\not{p}) \\
&= [(1-y-z)\not{q} + z\not{p}']\gamma^\mu(-y\not{q} + z\not{p}) \\
&= [(1-y-z)\not{q} + zm]\gamma^\mu(-y\not{q} + zm) \\
&= -y(1-y-z)\not{q}\gamma^\mu\not{q} + z(1-y-z)m\not{q}\gamma^\mu - yzm\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
&= -y(1-y-z)(2q^\mu - \gamma^\mu\not{q})\not{q} + z(1-y-z)m(2q^\mu - \gamma^\mu\not{q}) - yzm\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
&= -2y(1-y-z)q^\mu\not{q} + y(1-y-z)\gamma^\mu\not{q}\not{q} + 2z(1-y-z)mq^\mu - z(1-y-z)m\gamma^\mu\not{q} \\
&\quad - yzm\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
&= y(1-y-z)\gamma^\mu q^2 + 2z(1-y-z)mq^\mu - z(1-z)m\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
&= y(1-y-z)\gamma^\mu q^2 + 2z(1-y-z)mq^\mu - z(1-z)m(2p'^\mu - 2m\gamma^\mu) + z^2m^2\gamma^\mu \\
&= y(1-y-z)\gamma^\mu q^2 + 2z(1-y-z)mq^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu \\
&= yx\gamma^\mu q^2 + 2zxmq^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu.
\end{aligned}$$

他の項と併せて，分子は

$$\begin{aligned}
&= -\frac{1}{2}\gamma^\mu\ell^2 + yx\gamma^\mu q^2 + 2zxmq^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu \\
&\quad + m^2\gamma^\mu + 2mk^\mu + 2m^2\gamma^\mu - 2mp^\mu \\
&= \left[-\frac{\ell^2}{2} + (3 + 2z - z^2)m^2 + xyq^2\right]\gamma^\mu + 2zxmq^\mu - 2z(1-z)m(p^\mu + q^\mu) \\
&\quad + 2m(\ell^\mu - yq^\mu + zp^\mu) - 2mp^\mu \\
&= \left[-\frac{\ell^2}{2} + (3 + 2z - z^2)m^2 + xyq^2\right]\gamma^\mu + 2(z^2-1)mp^\mu + 2(zx - z + z^2 - y)mq^\mu \\
&= \left[-\frac{\ell^2}{2} + (3 + 2z - z^2)m^2 + xyq^2\right]\gamma^\mu + (z^2-1)m(p^\mu + p'^\mu) + (x-y)(1+z)mq^\mu \\
&= \left[-\frac{\ell^2}{2} + (3 + 2z - z^2)m^2 + xyq^2\right]\gamma^\mu + (z^2-1)m(p^\mu + p'^\mu).
\end{aligned}$$

Gordon 恒等式を使って変形すれば，

$$\begin{aligned}
&= \left[-\frac{\ell^2}{2} + (3 + 2z - z^2)m^2 + xyq^2\right]\gamma^\mu + (z^2-1)m(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) \\
&= \left[-\frac{\ell^2}{2} + (1+z)^2m^2 + xyq^2\right]\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}2m^2(1-z^2).
\end{aligned}$$

(6.33)(6.49) から

$$\begin{aligned}F_2(q^2) &= \frac{i\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{(\ell^2-\Delta)^3}\overline{u}(p')2m^2(1-z^2)u(p)\\ &= \frac{\lambda^2}{2}\frac{2m^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{1-z^2}{-xyq^2+(1-z)^2m^2+zm_h{}^2}.\end{aligned}$$

(6.37) から

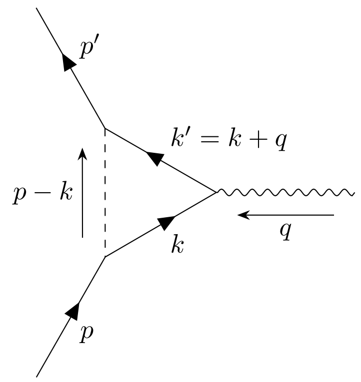
$$\begin{aligned}\frac{g-2}{2} &= F_2(q^2=0)\\ &= \frac{\lambda^2}{2}\frac{2m^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{1-z^2}{(1-z)^2m^2+zm_h{}^2}\\ &= \frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\frac{(1-z^2)(1-z)}{(1-z)^2+z(m_h/m)^2}.\end{aligned}$$

(c)

相互作用のハミルトニアンは

$$\mathcal{H}_{\text{int}} = \frac{i\lambda}{\sqrt{2}}a\overline{\psi}\gamma^5\psi$$

なので，これは vertex factor が $\lambda\gamma^5/\sqrt{2}$ の QED (p.123).



p.189 以降と同様に頂点補正を求める．上図に対応する振幅は

$$\begin{aligned}\overline{u}(p')\delta\Gamma^\mu(p,p')u(p) &= -\frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\gamma^5\left[\not{k}'\gamma^\mu\not{k}+m^2\gamma^\mu+m(\not{k}'\gamma^\mu+\gamma^\mu\not{k})\right]\gamma^5u(p)}{[(k-p)^2-m_a{}^2](k'^2-m^2)(k^2-m^2)}\\ &= \frac{i\lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\overline{u}(p')\left[\not{k}'\gamma^\mu\not{k}+m^2\gamma^\mu-m(\not{k}'\gamma^\mu+\gamma^\mu\not{k})\right]u(p)}{[(k-p)^2-m_a{}^2](k'^2-m^2)(k^2-m^2)}\end{aligned}$$

で与えられる（以降は両端のスピノル $\overline{u}(p')\cdots u(p)$ は省略する）.

まず，分母を求める．

$$\ell = k+yq-zp, \qquad D=\ell^2-\Delta+i\epsilon, \qquad \Delta = -xyq^2+(1-z)^2m^2+zm_a{}^2$$

とおけば，分母は

$$\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{D^3} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{(\ell^2-\Delta+i\epsilon)^3}.$$

分子は

$$\begin{aligned}
&= -\frac{1}{2}\gamma^\mu\ell^2 + yx\gamma^\mu q^2 + 2zxmq^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu \\
&\quad + m^2\gamma^\mu - 2mk^\mu - 2m^2\gamma^\mu + 2mp^\mu \\
&= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + 2zxmq^\mu - 2z(1-z)m(p^\mu+q^\mu) \\
&\quad - 2m(\ell^\mu - yq^\mu + zp^\mu) + 2mp^\mu \\
&= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + 2(z-1)^2mp^\mu + 2(zx - z + z^2 + y)mq^\mu \\
&= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + (z-1)^2m(p^\mu + p'^\mu) + (x-y)(z-1)mq^\mu \\
&= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + (z-1)^2m(p^\mu + p'^\mu) \\
&= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + (z-1)^2m(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) \\
&= \left[-\frac{\ell^2}{2} + (z-1)^2m^2 + xyq^2\right]\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(-2m^2)(z-1)^2.
\end{aligned}$$

(6.33)(6.49) から

$$\begin{aligned}
F_2(q^2) &= \frac{i\lambda^2}{2}\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{(\ell^2-\Delta)^3}\overline{u}(p')(-2m^2)(z-1)^2u(p) \\
&= -\frac{\lambda^2}{2}\frac{2m^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{(z-1)^2}{-xyq^2+(1-z)^2m^2+zm_a{}^2}.
\end{aligned}$$

(6.37) から

$$\begin{aligned}
\frac{g-2}{2} &= F_2(q^2=0) \\
&= -\frac{\lambda^2}{2}\frac{2m^2}{(4\pi)^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{(z-1^2)}{(1-z)^2m^2+zm_a{}^2} \\
&= -\frac{\lambda^2}{(4\pi)^2}\int_0^1 dz\frac{(1-z)^3}{(1-z)^2+z(m_a/m)^2}.
\end{aligned}$$

Chapter 7

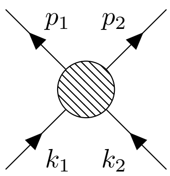
Radiative Corrections: Some Formal Developments

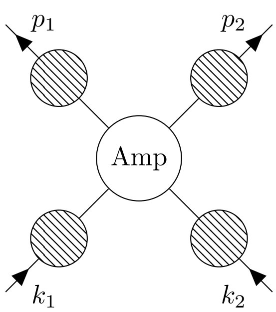
7.2 The LSZ Reduction formula

(7.45)

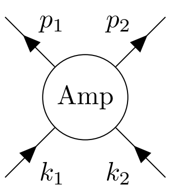
直前にある 4 点相関関数について証明する．LSZ 簡約公式 (7.42) から

$$\begin{aligned}
&\left(\prod_1^2 \frac{\sqrt{Z}i}{{p_i}^2 - m^2}\right)\left(\prod_1^2 \frac{\sqrt{Z}i}{{k_i}^2 - m^2}\right)\langle \boldsymbol{p}_1\boldsymbol{p}_2|\, S\,|\boldsymbol{k}_1\boldsymbol{k}_2\rangle \\
&= \left(\prod_1^2 \int d^4x_i e^{ip_i\cdot x_i}\right)\left(\prod_1^2 \int d^4y_i e^{ik_i\cdot y_i}\right)\langle\Omega|\, T\{\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\}\,|\Omega\rangle
\end{aligned}$$

$=$ 

$=$ 

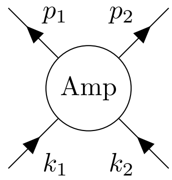
(7.44) とその後の式から

$=$ 

$$\times \frac{iZ}{{p_1}^2 - m^2}\frac{iZ}{{p_2}^2 - m^2}\frac{iZ}{{k_1}^2 - m^2}\frac{iZ}{{k_2}^2 - m^2}.$$

以上から，

$$\langle \boldsymbol{p}_1\boldsymbol{p}_2|\, S\, |\boldsymbol{k}_1\boldsymbol{k}_2\rangle = (\sqrt{Z})^4$$



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7.3 The Optical Theorem

(7.53)

(7.52) の積分

$$\begin{aligned} i\delta\mathcal{M} &= \frac{\lambda^2}{2}\int \frac{d^3q}{(2\pi)^4}\int dq^0 \frac{1}{(q-k/2)^2-m^2+i\epsilon}\frac{1}{(q+k/2)^2-m^2+i\epsilon} \\ &= \frac{\lambda^2}{2}\int \frac{d^3q}{(2\pi)^4}\int dq^0 \frac{1}{(q^0-k^0/2)^2-E_{\boldsymbol{q}}{}^2+i\epsilon}\frac{1}{(q^0+k^0/2)^2-E_{\boldsymbol{q}}{}^2+i\epsilon} \end{aligned}$$

を下半分の半円上で q^0 について実行する．この際に，$q^0 = -k^0/2 + E_{\boldsymbol{q}} - i\epsilon$ の極の留数のみ取れば，

$$\begin{aligned} i\delta\mathcal{M} &= \frac{\lambda^2}{2}\int \frac{d^3q}{(2\pi)^4}\frac{1}{(-k^0+E_{\boldsymbol{q}})^2-E_{\boldsymbol{q}}{}^2}\frac{-2\pi i}{2E_{\boldsymbol{q}}} \\ &= \frac{\lambda^2}{2}\int \frac{d^3q}{(2\pi)^4}\int dq^0 \frac{1}{(q^0-k^0/2)^2-E_{\boldsymbol{q}}{}^2}(-2\pi i)\,\delta((k/2+q)^2-m^2) \end{aligned}$$

となる．従って，極を限定する操作は

$$\frac{1}{(q+k/2)^2-m^2+i\epsilon} \to -2\pi i\,\delta((k/2+q)^2-m^2)$$

とみなすことに等しい．

(7.55)

まず，

$$\frac{1}{x+i\epsilon}-\frac{1}{x-i\epsilon} = \frac{-2i\epsilon}{x^2+\epsilon^2} = -2\pi i\,\delta(x)$$

である．実際，$a<0<b$ で積分すれば，

$$\int_a^b \frac{1}{x^2+\epsilon^2}\,dx = \frac{1}{\epsilon}\int_{a/\epsilon}^{b/\epsilon}\frac{1}{x^2+1}\,dx \approx \frac{1}{\epsilon}\int_{-\infty}^{+\infty}\frac{1}{x^2+1}\,dx = \frac{\pi}{\epsilon}$$

となる．$[a,b]$ が 0 を含まなければ積分は 0 となる．

(7.51) のあとで議論したように，$s\pm i\epsilon$ での $\mathcal{M}$ の不連続性を計算する．ϵ は微小なので，$k^0\pm i\epsilon$ で計算してもよい．先ほど得られた式

$$i\delta\mathcal{M}(k^0) = \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{(k^0/2-q^0)^2-E_{\boldsymbol{q}}{}^2}(-2\pi i)\,\delta((k/2+q)^2-m^2)$$

を使えば，

$$i\,\mathrm{Disc}\,\mathcal{M} = i\delta\mathcal{M}(k^0+i\epsilon) - i\delta\mathcal{M}(k^0-i\epsilon)$$

$$
\begin{aligned}
&= \frac{\lambda^2}{2}\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{(k^0/2-q^0+i\epsilon/2)^2-E_{\boldsymbol{q}}{}^2}-\frac{1}{(k^0/2-q^0-i\epsilon/2)^2-E_{\boldsymbol{q}}{}^2}\right] \\
&\qquad\times(-2\pi i)\,\delta((k/2+q)^2-m^2) \\
&= \frac{\lambda^2}{2}\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{(k^0/2-q^0+i\epsilon/2)^2-E_{\boldsymbol{q}}{}^2}-\frac{1}{(k^0/2-q^0-i\epsilon/2)^2-E_{\boldsymbol{q}}{}^2}\right] \\
&\qquad\times(-2\pi i)\,\delta((k/2+q)^2-m^2) \\
&= \frac{\lambda^2}{2}\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{(k^0/2-q^0)^2-E_{\boldsymbol{q}}{}^2+i\epsilon}-\frac{1}{(k^0/2-q^0)^2-E_{\boldsymbol{q}}{}^2-i\epsilon/2}\right] \\
&\qquad\times(-2\pi i)\,\delta((k/2+q)^2-m^2) \\
&= \frac{\lambda^2}{2}\int\frac{d^4q}{(2\pi)^4}(-2\pi i)\,\delta((k^0/2-q^0)^2-E_{\boldsymbol{q}}{}^2)(-2\pi i)\,\delta((k/2+q)^2-m^2) \\
&= \frac{\lambda^2}{2}\int\frac{d^4q}{(2\pi)^4}(-2\pi i)\,\delta((k/2-q)^2-m^2)(-2\pi i)\,\delta((k/2+q)^2-m^2)
\end{aligned}
$$

となる．従って，不連続値を求める操作は

$$
\frac{1}{(q-k/2)^2-m^2+i\epsilon}\to -2\pi i\,\delta((k/2-q)^2-m^2)
$$

とみなすことに等しい．

(7.58)〜(7.59)

m の定義は

$$
m^2-m_0{}^2-\operatorname{Re}M^2(m^2)=0.
$$

(7.44) と同様に，修正した伝播函数は

$$
\frac{i}{p^2-m_0{}^2-\operatorname{Re}M^2(p^2)}\sim\frac{iZ}{p^2-m^2}
$$

で与えられる．従って，

$$
\begin{aligned}
\frac{i}{p^2-m_0{}^2-M^2(p^2)} &= \frac{i}{p^2-m_0{}^2-\operatorname{Re}M^2(p^2)-i\operatorname{Im}M^2(p^2)} \\
&= \frac{iZ}{p^2-m^2-iZ\operatorname{Im}M^2(p^2)}.
\end{aligned}
$$

Problems

Problem 6.3(a) の Yukawa vertex factor を求める際に dimensional regularization をする．これにより，$\not\ell\gamma^\mu\not\ell$ の計算結果が変化する．(6.46) から

$$
\not\ell\gamma^\mu\not\ell=\ell_a\ell_b\gamma^a\gamma^\mu\gamma^b\to\frac{\ell^2}{d}g_{ab}\gamma^a\gamma^\mu\gamma^b=\frac{\ell^2}{d}\gamma^a\gamma^\mu\gamma_a=\frac{2-d}{d}\ell^2\gamma^\mu.
$$

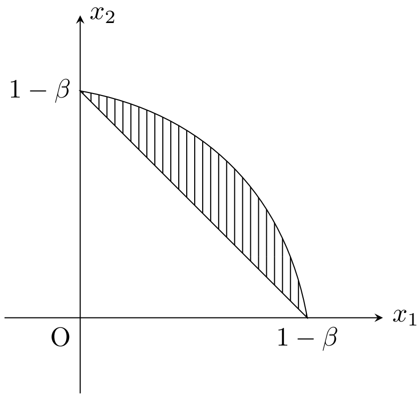
Final Project I: Radiation of Gluon Jets

(d)

$\mu^2/s = \beta$ とする．被積分函数は

$$\frac{{x_1}^2 + {x_2}^2}{(1-x_1)(1-x_2)} + \left[\frac{2(x_1+x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2}\right]\beta + \frac{2}{(1-x_1)(1-x_2)}\beta^2.$$

積分範囲は (b) で求めた次の通り．



β の 1 乗以上の項は無視する．被積分函数のうち β を含まない項は

$$\begin{aligned}
I_0 &= \int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2\, \frac{{x_1}^2 + {x_2}^2}{(1-x_1)(1-x_2)} \\
&= \int \frac{dx_1}{1-x_1} \int \frac{{x_2}^2 + {x_1}^2}{1-x_2}\, dx_2 \\
&= \int \frac{dx_1}{1-x_1} \int \left(-x_2 - 1 - \frac{{x_1}^2+1}{x_2-1}\right) dx_2 \\
&= \int \frac{dx_1}{1-x_1} \frac{1}{2} \left[-2({x_1}^2+1)\log(1-x_2) - x_2(x_2+2)\right]_{1-x_1-\beta}^{1-\beta/(1-x_1)} \\
&= \int \frac{dx_1}{1-x_1} \frac{1}{2} \left[-2({x_1}^2+1)\log\frac{\beta}{1-x_1} + 2({x_1}^2+1)\log(x_1+\beta)\right] \\
&\quad + \int \frac{dx_1}{1-x_1} \frac{1}{2} \left[-\left(1 - \frac{\beta}{1-x_1}\right)\left(3 - \frac{\beta}{1-x_1}\right) + (1-x_1-\beta)(3-x_1-\beta)\right] \\
&\approx \int \frac{dx_1}{1-x_1} \left[({x_1}^2+1)\log(1-x_1) - ({x_1}^2+1)\log\beta + ({x_1}^2+1)\log(x_1+\beta)\right] \\
&\quad + \int \frac{dx_1}{1-x_1} \frac{{x_1}^2 - 4x_1}{2} + \int \frac{dx_1}{1-x_1} \left[\frac{2\beta}{1-x_1} - \frac{1}{2}\frac{\beta^2}{(1-x_1)^2}\right].
\end{aligned}$$

ここで，$(4-2x_1)\beta+\beta^2$ は積分すれば β^1 以上の次数となるため，無視した．計算を継続する：

$$
\begin{aligned}
&= \int_0^{1-\beta} \frac{{x_1}^2+1}{1-x_1} \log(1-x_1)\,dx_1 - \log\beta \int_0^{1-\beta} \frac{{x_1}^2+1}{1-x_1}\,dx_1 \\
&\quad + \int_0^{1-\beta} \frac{{x_1}^2+1}{1-x_1} \log(x_1+\beta)\,dx_1 + \frac{1}{2}\int_0^{1-\beta} \frac{{x_1}^2-4x_1}{1-x_1}\,dx_1 \\
&\quad + \int_0^{1-\beta} \left[\frac{2\beta}{(1-x_1)^2} - \frac{1}{2}\frac{\beta^2}{(1-x_1)^3}\right] dx_1.
\end{aligned}
$$

第 1 項：

$$
\begin{aligned}
\int_0^{1-\beta} \frac{{x_1}^2+1}{1-x_1} \log(1-x_1)\,dx_1 &= \int_\beta^1 \frac{x^2-2x+2}{x} \log x\,dx \\
&= \left[2x - \frac{x^2}{4} + \frac{x(x-4)}{2}\log x + \log^2 x\right]_\beta^1 \\
&\approx \frac{7}{4} - \log^2\beta.
\end{aligned}
$$

第 2 項：

$$
\begin{aligned}
\log\beta \int_0^{1-\beta} \frac{{x_1}^2+1}{1-x_1}\,dx_1 &= \log\beta \int_\beta^1 \frac{x^2-2x+2}{x}\,dx \\
&= \log\beta \left[\frac{x^2}{2} - 2x + 2\log x\right]_\beta^1 \\
&\approx -\frac{3}{2}\log\beta - 2\log^2\beta.
\end{aligned}
$$

第 3 項：

$$
\begin{aligned}
&\int_0^{1-\beta} \frac{{x_1}^2+1}{1-x_1} \log(x_1+\beta)\,dx_1 \\
&= \int_\beta^1 \frac{x^2-2x+2}{x} \log(1+\beta-x)\,dx \\
&= \left[\frac{(1+\beta-x)^2}{2}\log(1+\beta-x)\right]_\beta^1 + (1+\beta)\int_\beta^1 \log(1+\beta-x)\,dx + \frac{1}{2}\int_\beta^1 (1+\beta-x)\,dx \\
&\quad - 2\int_\beta^1 \log(1+\beta-x)\,dx \\
&\quad + 2\int_\beta^1 \frac{\log(1+\beta-x)}{x}\,dx \\
&\approx -\int_\beta^1 \log(1+\beta-x)\,dx + \frac{1}{2}\int_\beta^1 (1+\beta-x)\,dx + 2\int_\beta^1 \frac{\log(1+\beta-x)}{x}\,dx \\
&= \left[(1+\beta-x)\log(1+\beta-x) + x - \frac{1}{4}(1+\beta-x)^2\right]_\beta^1 + 2\int_\beta^1 \frac{\log(1+\beta-x)}{x}\,dx \\
&\approx \frac{5}{4} + 2\int_\beta^1 \frac{\log(1+\beta-x)}{x}\,dx \\
&\approx \frac{5}{4} + 2\int_{\beta/(1+\beta)}^{1/(1+\beta)} \frac{\log(1+\beta)+\log(1-x)}{x}\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\approx \frac{5}{4} + 2\int_{\beta/(1+\beta)}^{1/(1+\beta)} \frac{\log(1-x)}{x}\,dx \\
&\approx \frac{5}{4} + 2\int_0^1 \frac{\log(1-x)}{x}\,dx \\
&= \frac{5}{4} - \frac{\pi^2}{3}.
\end{aligned}
$$

第 4 項：

$$
\begin{aligned}
\frac{1}{2}\int_0^{1-\beta} \frac{{x_1}^2 - 4x_1}{1-x_1}\,dx_1 &= \frac{1}{2}\int_0^{1-\beta} \left(3 - x - \frac{3}{1-x}\right)\,dx \\
&= \frac{1}{2}\left[3x - \frac{x^2}{2} + 3\log(1-x)\right]_0^{1-\beta} \\
&= \frac{5}{4} + \frac{3}{2}\log\beta.
\end{aligned}
$$

第 5 項：

$$
\begin{aligned}
\int_0^{1-\beta} \left[\frac{2\beta}{(1-x_1)^2} - \frac{1}{2}\frac{\beta^2}{(1-x_1)^3}\right]\,dx_1 &= \left[\frac{2\beta}{1-x_1} - \frac{1}{4}\frac{\beta^2}{(1-x_1)^2}\right]_0^{1-\beta} \\
&\approx \frac{7}{4}.
\end{aligned}
$$

以上から，

$$
\begin{aligned}
I_0 &= \frac{7}{4} - \log^2\beta - \left(-\frac{3}{2}\log\beta - 2\log^2\beta\right) + \frac{5}{4} - \frac{\pi^2}{3} + \frac{5}{4} + \frac{3}{2}\log\beta + \frac{7}{4} \\
&= 6 + 3\log\beta + \log^2\beta - \frac{\pi^2}{3}.
\end{aligned}
\tag{I.1}
$$

被積分函数のうち β^1 を含む項は

$$
\begin{aligned}
I_1 &= \beta\int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 \left[\frac{2(x_1+x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2}\right] \\
&= \beta\int dx_1 \int dx_2 \left[\frac{2(x_1+x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2}\right] \\
&= \beta\int dx_1 \int dx_2 \left[\frac{4 - 2(1-x_1) - 2(1-x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2}\right] \\
&= \beta\int dx_1 \int dx_2 \left[\frac{4}{(1-x_1)(1-x_2)} - \frac{2}{1-x_1} - \frac{2}{1-x_2} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2}\right] \\
&= \beta\int dx_1 \int dx_2 \left[\frac{4}{(1-x_1)(1-x_2)} - \frac{4}{1-x_1} - \frac{2}{(1-x_1)^2}\right],
\end{aligned}
$$

計算の最後で，積分が x_1, x_2 に関して対称であることを用いた．第 1 項：

$$
\begin{aligned}
&4\beta\int_0^{1-\beta} \frac{dx_1}{1-x_1} \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} \frac{dx_2}{1-x_2} \\
&= 4\beta\int_0^{1-\beta} \left[-\log(1-x_2)\right]_{1-x_1-\beta}^{1-\beta/(1-x_1)} \\
&= 4\beta\int_0^{1-\beta} \left[\log(x_1+\beta) - \log\frac{\beta}{1-x_1}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= 4\beta \int_0^{1-\beta} \frac{\log(x_1+\beta)}{1-x_1}dx_1 - 4\beta\log\beta \int_0^{1-\beta} \frac{dx_1}{1-x_1} + 4\beta \int_0^{1-\beta} \frac{\log(1-x_1)}{1-x_1}dx_1 \\
&= 4\beta \int_\beta^1 \frac{\log(1+\beta-x)}{x}dx + 4\beta\log\beta \left[\log(1-x_1)\right]_0^{1-\beta} + 2\beta\left[\log^2(1-x_1)\right]_0^{1-\beta} \\
&= 4\beta \int_\beta^1 \frac{\log(1+\beta-x)}{x}dx + 4\beta\log^2\beta + 2\beta\log^2\beta
\end{aligned}
$$

初めの積分については，I_0 の第 3 項と同様に計算すればよいので，

$$
\begin{aligned}
&\approx 4\beta\left(-\frac{\pi^2}{6}\right) + 4\beta\log^2\beta + 2\beta\log^2\beta \\
&\approx 0.
\end{aligned}
$$

第 2 項：

$$
\begin{aligned}
4\beta \int_0^{1-\beta} \frac{dx_1}{1-x_1} \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 &= 4\beta \int_0^{1-\beta} \frac{dx_1}{1-x_1}\left(x_1+\beta-\frac{\beta}{1-x_1}\right) \\
&= 4\beta \int_0^{1-\beta} dx_1 \left[-1+\frac{\beta+1}{1-x_1}-\frac{\beta}{(1-x_1)^2}\right] \\
&\approx 0.
\end{aligned}
$$

第 3 項：

$$
\begin{aligned}
2\beta \int_0^{1-\beta} \frac{dx_1}{(1-x_1)^2} \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 &= 2\beta \int_0^{1-\beta} \frac{dx_1}{(1-x_1)^2}\left(x_1+\beta-\frac{\beta}{1-x_1}\right) \\
&= 2\beta \int_0^{1-\beta} dx_1 \left[\frac{1}{x_1-1}+\frac{\beta+1}{(1-x_1)^2}-\frac{\beta}{(1-x_1)^3}\right] \\
&= 2\beta \left[\log(1-x_1)+\frac{\beta+1}{1-x_1}-\frac{\beta}{2(1-x_1)^2}\right]_0^{1-\beta} \\
&= 2\beta\left[\log\beta+\frac{1}{\beta}-\beta-\frac{1}{2\beta}+\frac{\beta}{2}\right] \\
&\approx 1.
\end{aligned}
$$

以上から，

$$
I_1 \approx 0-0-1=-1. \tag{I.2}
$$

I_1 の第 1 項の計算結果から，積分のうち β^2 を含む項は 0 である．[I.1][I.2] から，積分は

$$
I_0+I_1 = 5+3\log\beta+\log^2\beta-\frac{\pi^2}{3}.
$$

(e)

(a) で得られた Feynman パラメータの積分

$$
\int_0^1 dx \int_0^{1-x} dz \left[\log\frac{z\beta}{z\beta-x(1-x-z)}+\frac{(1-x)(x+z)}{z\beta-x(1-x-z)}\right]
$$

を計算する．

発散しない項：

$$
\begin{aligned}
\int_0^1 dx \int_0^{1-x} dz \log(z\beta) &= \int_0^1 dx \int_0^{1-x} dz \,(\log z + \log\beta) \\
&= \int_0^1 dx \left[z\log z - z + z\log\beta\right]_0^{1-x} \\
&= \int_0^1 dx \left[(1-x)\log(1-x) - (1-x) + (1-x)\log\beta\right] \\
&= \int_0^1 dx \left[x\log x - x + x\log\beta\right] \\
&= \left[\frac{x^2}{2}\log x - \frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{2}\log\beta\right]_0^1 \\
&= \frac{1}{2}\log\beta - \frac{3}{4}.
\end{aligned}
\tag*{[I.3]}
$$

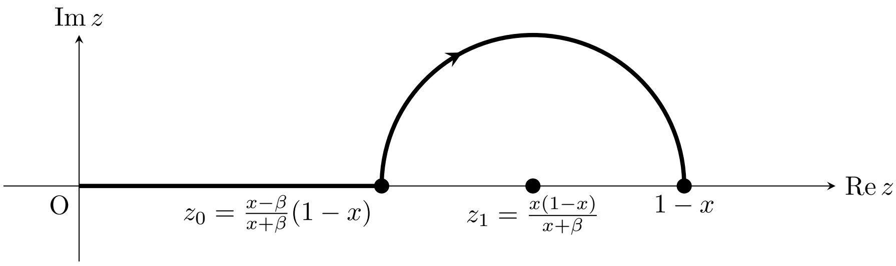
簡単のため，

$$
F(z) = z\beta - x(1-x-z) = (x+\beta)z - x(1-x)
$$

とおく．$\log F(z)$ には特異点

$$
z_1 = \frac{x-\beta}{x+\beta}(1-x)
$$

が存在するので，それを回避するように積分する．



線分での積分は[*1]

$$
\begin{aligned}
&\int_{F=-x(1-x)}^{F=-\beta(1-x)} dz \log F(z) \\
&= \int_{-x(1-x)}^{-\beta(1-x)} dF \,\frac{dz}{dF} \log F \\
&= \frac{1}{x+\beta}\int_{\beta(1-x)}^{x(1-x)} dr\,(\log r + i\pi) \\
&= i\pi\frac{x-\beta}{x+\beta}(1-x) + \frac{1}{x+\beta}\int_{\beta(1-x)}^{x(1-x)} \log r \, dr \\
&= i\pi\frac{x-\beta}{x+\beta}(1-x) + \frac{1}{x+\beta}\left[r\log r - r\right]_{\beta(1-x)}^{x(1-x)} \\
&= i\pi\frac{x-\beta}{x+\beta}(1-x) + \frac{x(1-x)}{x+\beta}\left[\log x(1-x) - 1\right] - \beta\frac{1-x}{x+\beta}\left[\log\beta(1-x) - 1\right]
\end{aligned}
$$

[*1] 積分結果のうち実部しか使わないので，分岐はどう選んでも結構

$$
\begin{aligned}
&= i\pi\frac{x-\beta}{x+\beta}(1-x) + \frac{x(1-x)}{x+\beta}\left[\log x + \log(1-x) - 1\right] - \beta\frac{1-x}{x+\beta}\left[\log\beta + \log(1-x) - 1\right] \\
&= (i\pi-1)\frac{x-\beta}{x+\beta}(1-x) + \frac{x-\beta}{x+\beta}(1-x)\log(1-x) + \frac{x(1-x)}{x+\beta}\log x - \beta\log\beta\frac{1-x}{x+\beta}.
\end{aligned}
$$

半円

$$
z = z_1 + \frac{1-x}{x+\beta}\beta e^{i\theta} \quad (\theta\colon \pi\to 0), \quad F = (x+\beta)(z-z_1) = \beta(1-x)e^{i\theta}
$$

での積分は

$$
\begin{aligned}
\int_{\theta=\pi}^{\theta=0} dz\log F &= \int_\pi^0 d\theta\,\frac{dz}{d\theta}\log\beta(1-x)e^{i\theta} \\
&= i\beta\frac{1-x}{x+\beta}\int_\pi^0 d\theta\, e^{i\theta}[i\theta + \log\beta(1-x)] \\
&= \beta\frac{1-x}{x+\beta}\int_0^\pi d\theta\,\theta e^{i\theta} + i\beta\frac{1-x}{x+\beta}\log\beta(1-x)\int_\pi^0 d\theta\, e^{i\theta} \\
&= \beta\frac{1-x}{x+\beta}\left[(1-i\theta)e^{i\theta}\right]_0^\pi + i\beta\frac{1-x}{x+\beta}\log\beta(1-x)\left[-ie^{i\theta}\right]_\pi^0 \\
&= \beta\frac{1-x}{x+\beta}\left[(1-i\theta)e^{i\theta}\right]_0^\pi + i\beta\frac{1-x}{x+\beta}\log\beta(1-x)\left[-ie^{i\theta}\right]_\pi^0 \\
&= \beta\frac{1-x}{x+\beta}[i\pi-2] + i\beta\frac{1-x}{x+\beta}\log\beta(1-x)[-2i] \\
&= (i\pi-2+2\log\beta)\beta\frac{1-x}{x+\beta} + 2\beta\frac{1-x}{x+\beta}\log(1-x)
\end{aligned}
$$

従って，積分は

$$
\begin{aligned}
\int_0^{1-x} dz\,\log\frac{1}{F} &= -(i\pi-1)\frac{x-\beta}{x+\beta}(1-x) - \frac{x-\beta}{x+\beta}(1-x)\log(1-x) - \frac{x(1-x)}{x+\beta}\log x + \beta\log\beta\frac{1-x}{x+\beta} \\
&\quad - (i\pi-2+2\log\beta)\beta\frac{1-x}{x+\beta} - 2\beta\frac{1-x}{x+\beta}\log(1-x) \\
&= -(i\pi-2+\log\beta)\beta\frac{1-x}{x+\beta} - (i\pi-1)\frac{x-\beta}{x+\beta}(1-x) \\
&\quad - (1-x)\log(1-x) - \frac{x(1-x)}{x+\beta}\log x.
\end{aligned}
$$

これを $x\colon 0\to 1$ で積分する．第 1 項：

$$
\begin{aligned}
-(i\pi-2+\log\beta)\beta\int_0^1 dx\,\frac{1-x}{x+\beta} &= -(i\pi-2+\log\beta)\beta\int_0^1 dx\,\left(-1+\frac{1+\beta}{x+\beta}\right) \\
&= -(i\pi-2+\log\beta)\beta\left[-x+(1+\beta)\log(x+\beta)\right]_0^1 \\
&= -(i\pi-2+\log\beta)\beta\left[-1+(1+\beta)\log\frac{1+\beta}{\beta}\right] \\
&\approx 0.
\end{aligned}
$$

第 2 項：

$$
\begin{aligned}
-(i\pi-1)\int_0^1 dx\,\frac{x-\beta}{x+\beta}(1-x) &= -(i\pi-1)\int_0^1 dx\,\frac{-(x+\beta)(x-1-2\beta)-2\beta(1+\beta)}{x+\beta} \\
&= -(i\pi-1)\int_0^1 dx\,\left[-x+1+2\beta-2\beta(1+\beta)\frac{1}{x+\beta}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= -(i\pi - 1)\left[-\frac{x^2}{2} + (1+2\beta)x - 2\beta(1+\beta)\log(x+\beta)\right]_0^1 \\
&= -(i\pi - 1)\left[-\frac{1}{2} + (1+2\beta) - 2\beta(1+\beta)\log\frac{1+\beta}{\beta}\right] \\
&= -(i\pi - 1)\left[-\frac{1}{2} + (1+2\beta) - 2\beta(1+\beta)\log\frac{1+\beta}{\beta}\right] \\
&\approx \frac{1}{2}(1 - i\pi).
\end{aligned}
$$

第 3 項：

$$
\begin{aligned}
-\int_0^1 dx\,(1-x)\log(1-x) &= -\int_0^1 dx\,x\log x \\
&= -\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right]_0^1 \\
&= \frac{1}{4}.
\end{aligned}
$$

第 4 項：

$$
\begin{aligned}
-\int_0^1 dx\,\frac{x(1-x)}{x+\beta}\log x &= \int_0^1 dx\left[x - \beta - 1 + \frac{\beta(\beta+1)}{x+\beta}\right]\log x \\
&= \int_0^1 x\log x\,dx - (\beta+1)\int_0^1 \log x\,dx + \beta(\beta+1)\int_0^1 \frac{\log x}{x+\beta}\,dx \\
&= -\frac{1}{4} + (\beta+1) + \beta(\beta+1)\int_0^1 \frac{\log x}{x+\beta}\,dx \\
&\approx \frac{3}{4}.
\end{aligned}
$$

以上から，

$$
\int_0^1 dx\int_0^{1-x} dz\,\log\frac{1}{F} = 0 + \frac{1}{2}(1-i\pi) + \frac{1}{4} + \frac{3}{4} = \frac{3-i\pi}{2}. \tag{I.4}
$$

有理多項式の項も極 z_1 が存在するので，先程と同じ経路で積分する．線分での積分は

$$
\begin{aligned}
\int_0^{z_0} dz\,\frac{(1-x)(x+z)}{z\beta - x(1-x-z)} &= \frac{1-x}{x+\beta}\int_0^{z_0}\frac{z+x}{z-z_1}\,dz \\
&= \frac{1-x}{x+\beta}\int_0^{z_0}\left(1 + \frac{x+z_1}{z-z_1}\right)dz \\
&= \frac{1-x}{x+\beta}\left[z_0 + (x+z_1)\log\left|\frac{z_0-z_1}{z_1}\right|\right] \\
&= \frac{1-x}{x+\beta}\left[\frac{x-\beta}{x+\beta}(1-x) + (1+\beta)\frac{x}{x+\beta}\log\frac{\beta}{x}\right] \\
&= \left(\frac{1-x}{x+\beta}\right)^2(x-\beta) + (1+\beta)\log\beta\frac{x(1-x)}{(x+\beta)^2} - (1+\beta)\frac{x(1-x)}{(x+\beta)^2}\log x.
\end{aligned}
$$

半円

$$
z = z_1 + \frac{1-x}{x+\beta}\beta e^{i\theta} \quad (\theta\colon \pi \to 0)
$$

での積分は

$$
\begin{aligned}
\int dz\,\frac{(1-x)(x+z)}{z\beta - x(1-x-z)} &= \frac{1-x}{x+\beta}\int_{\theta=\pi}^{\theta=0}\frac{z+x}{z-z_1}\frac{dz}{d\theta}\,d\theta \\
&= \frac{1-x}{x+\beta}\int_{\pi}^{0} i(z+x)\,d\theta \\
&= i\frac{1-x}{x+\beta}\int_{\pi}^{0}\left[(1+\beta)\frac{x}{x+\beta} + \beta\frac{1-x}{x+\beta}e^{i\theta}\right]d\theta \\
&= i\frac{1-x}{x+\beta}\left[-\pi(1+\beta)\frac{x}{x+\beta} - 2i\beta\frac{1-x}{x+\beta}\right] \\
&= -i\pi(1+\beta)\frac{x(1-x)}{(x+\beta)^2} + 2\beta\left(\frac{1-x}{x+\beta}\right)^2.
\end{aligned}
$$

以上から，

$$
\begin{aligned}
&\int_0^{1-x} dz\,\frac{(1-x)(x+z)}{z\beta - x(1-x-z)} \\
&= \left(\frac{1-x}{x+\beta}\right)^2(x-\beta) + (1+\beta)\log\beta\frac{x(1-x)}{(x+\beta)^2} - (1+\beta)\frac{x(1-x)}{(x+\beta)^2}\log x \\
&\quad - i\pi(1+\beta)\frac{x(1-x)}{(x+\beta)^2} + 2\beta\left(\frac{1-x}{x+\beta}\right)^2 \\
&= \frac{(1-x)^2}{x+\beta} + (1+\beta)(\log\beta - i\pi)\frac{x(1-x)}{(x+\beta)^2} - (1+\beta)\frac{x(1-x)}{(x+\beta)^2}\log x.
\end{aligned}
$$

これを $x\colon 0\to 1$ で積分する．第 1 項：

$$
\begin{aligned}
\int_0^1 dx\,\frac{(1-x)^2}{x+\beta} &= \int_\beta^{1+\beta}\frac{(1+\beta-x)^2}{x} \\
&= \int_\beta^{1+\beta}\frac{x^2 - 2(1+\beta)x + (1+\beta)^2}{x} \\
&= \left[\frac{x^2}{2} - 2(1+\beta)x + (1+\beta)^2\log x\right]_\beta^{1+\beta} \\
&= \frac{1+2\beta}{2} - 2(1+\beta) + (1+\beta)^2\log\frac{1+\beta}{\beta} \\
&\approx -\frac{3}{2} - \log\beta.
\end{aligned}
$$

第 2 項：

$$
\begin{aligned}
(1+\beta)(\log\beta - i\pi)\int_0^1 dx\,\frac{x(1-x)}{(x+\beta)^2} &= (1+\beta)(\log\beta - i\pi)\int_\beta^{1+\beta}\frac{(x-\beta)(1+\beta-x)}{x^2} \\
&= (1+\beta)(\log\beta - i\pi)\int_\beta^{1+\beta}\frac{-x^2 + (1+2\beta)x - \beta(1+\beta)}{x^2} \\
&= (1+\beta)(\log\beta - i\pi)\left[-x + (1+2\beta)\log x + \beta(1+\beta)\frac{1}{x}\right]_\beta^{1+\beta} \\
&= (1+\beta)(\log\beta - i\pi)\left[-1 + (1+2\beta)\log\frac{1+\beta}{\beta} - 1\right] \\
&\approx (\log\beta - i\pi)\left[-2 - \log\beta\right]
\end{aligned}
$$

$$= -2\log\beta - \log^2\beta + i\pi(2+\log\beta)$$

第 3 項：

$$\begin{aligned}
&-\int_0^1 dx\,\frac{x(1-x)}{(x+\beta)^2}\log x\\
&= -\int_\beta^{1+\beta} dx\,\frac{(x-\beta)(1+\beta-x)}{x^2}\log(x-\beta)\\
&= -\int_\beta^{1+\beta} dx\,\frac{-x^2+(1+2\beta)x-\beta(1+\beta)}{x^2}\log(x-\beta)\\
&= \int_\beta^{1+\beta} dx\log(x-\beta) - (1+2\beta)\int_\beta^{1+\beta} dx\,\frac{\log(x-\beta)}{x} + \beta(1+\beta)\int_\beta^{1+\beta} dx\,\frac{\log(x-\beta)}{x^2}.
\end{aligned}$$

第 3 項の第 1 項：

$$\int_\beta^{1+\beta} dx\log(x-\beta) = \int_0^1 dx\log x = -1.$$

第 3 項の第 2 項：

$$\begin{aligned}
\int_\beta^{1+\beta} dx\,\frac{\log(x-\beta)}{x} &= \int_1^{1+1/\beta} dx\,\frac{\log\beta(x-1)}{x}\\
&= \int_1^{1+1/\beta} dx\,\frac{\log\beta+\log(x-1)}{x}\\
&= \log\beta\int_1^{1+1/\beta}\frac{dx}{x} + \int_1^{1+1/\beta} dx\,\frac{\log(x-1)}{x}\\
&= \log\beta\log\frac{1+\beta}{\beta} + \int_1^{\beta/(1+\beta)}\left(-\frac{dx}{x^2}\right)x\log\left(\frac{1}{x}-1\right)\\
&\approx -\log^2\beta + \int_{\beta/(1+\beta)}^1 dx\,\frac{\log(1-x)-\log x}{x}\\
&= -\log^2\beta + \int_{\beta/(1+\beta)}^1 dx\,\frac{\log(1-x)}{x} - \int_{\beta/(1+\beta)}^1 dx\,\frac{\log x}{x}\\
&\approx -\log^2\beta - \frac{\pi^2}{6} - \left[\frac{1}{2}\log^2 x\right]_{\beta/(1+\beta)}^1\\
&\approx -\log^2\beta - \frac{\pi^2}{6} + \frac{1}{2}\log^2\frac{\beta}{1+\beta}\\
&\approx -\frac{1}{2}\log^2\beta - \frac{\pi^2}{6}.
\end{aligned}$$

第 3 項の第 3 項：

$$\begin{aligned}
\beta(1+\beta)\int_\beta^{1+\beta} dx\,\frac{\log(x-\beta)}{x^2} &= (1+\beta)\int_1^{1+1/\beta} dx\,\frac{\log\beta(x-1)}{x^2}\\
&= (1+\beta)\int_1^{1+1/\beta} dx\,\frac{\log\beta+\log(x-1)}{x^2}\\
&= (1+\beta)\log\beta\int_1^{1+1/\beta}\frac{dx}{x^2} + (1+\beta)\int_1^{1+1/\beta} dx\,\frac{\log(x-1)}{x^2}
\end{aligned}$$

$$
\begin{aligned}
&= \log\beta + (1+\beta)\int_1^{\beta/(1+\beta)} \left(-\frac{dx}{x^2}\right) x^2 \log\left(\frac{1}{x}-1\right) \\
&= \log\beta + (1+\beta)\int_{\beta/(1+\beta)}^{1} dx \log\frac{1-x}{x} \\
&= \log\beta + (1+\beta)\int_{\beta/(1+\beta)}^{1} dx \log(1-x) - (1+\beta)\int_{\beta/(1+\beta)}^{1} dx \log x \\
&= \log\beta + (1+\beta)\int_{0}^{1/(1+\beta)} dx \log x - (1+\beta)\int_{\beta/(1+\beta)}^{1} dx \log x \\
&= \log\beta + (1+\beta)[x\log x - x]_0^{1/(1+\beta)} - (1+\beta)[x\log x - x]_{\beta/(1+\beta)}^{1} \\
&\approx \log\beta.
\end{aligned}
$$

従って，第 3 項は

$$
-\int_0^1 dx\, \frac{x(1-x)}{(x+\beta)^2}\log x = -1 + \frac{1}{2}\log^2\beta + \frac{\pi^2}{6} + \log\beta.
$$

以上から，x での積分結果は

$$
\begin{aligned}
&\int_0^{1-x} dz\, \frac{(1-x)(x+z)}{z\beta - x(1-x-z)} \\
&= -\frac{3}{2} - \log\beta - 2\log\beta - \log^2\beta + i\pi(2+\log\beta) - 1 + \frac{1}{2}\log^2\beta + \frac{\pi^2}{6} + \log\beta \qquad \text{[I.5]} \\
&= -\frac{5}{2} - \frac{1}{2}\log^2\beta - 2\log\beta + \frac{\pi^2}{6} + i\pi(2+\log\beta).
\end{aligned}
$$

[I.3][I.4][I.5] から，積分は

$$
\begin{aligned}
&\frac{1}{2}\log\beta - \frac{3}{4} + \frac{3-i\pi}{2} - \frac{5}{2} - \frac{1}{2}\log^2\beta - 2\log\beta + \frac{\pi^2}{6} + i\pi(2+\log\beta) \\
&= -\frac{7}{4} - \frac{1}{2}\log^2\beta - \frac{3}{2}\log\beta + \frac{\pi^2}{6} + \frac{i\pi}{2}(3+2\log\beta).
\end{aligned}
$$

Part II

Renormalization

Chapter 9

Functional Methods

9.5 Functional Quantization of Spinor Fields

Functional determinant

$i\not{D} - m$ の固有値と固有ベクトルを

$$(i\not{D} - m)\psi_i = b_i\psi_i$$

とする．(9.76) を Wick 回転して

$$\begin{aligned}
\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left[-\int d^4x\,\bar{\psi}(i\not{D} - m)\psi\right] &= \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \exp\left[-\sum_i \int d^4x\,\bar{\psi}_i b_i \psi_i\right] \\
&= \prod_i \int \mathcal{D}\bar{\psi}_i\,\mathcal{D}\psi_i \exp\left[-\int d^4x\,\bar{\psi}_i b_i \psi_i\right] \\
&= \prod_i \prod_x \int d\bar{\psi}_i(x)\,d\psi_i(x) \exp\left[-\bar{\psi}_i(x) b_i(x) \psi_i(x)\right] \\
&= \prod_i \prod_x b_i(x).
\end{aligned}$$

最後に (9.67) を使った．これは $i\not{D} - m$ の固有値の積なので，$\det(i\not{D} - m)$ と表せる．

Problems

Problem 9.1: Scalar QED

伝播函数を計算しておく．スカラー粒子は

$$\begin{aligned}
(-\partial^2 - m^2 + i\epsilon)D_F(x-y) &= (-\partial^2 - m^2 + i\epsilon)\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2 + i\epsilon}e^{-ip(x-y)} \\
&= i\int \frac{d^4p}{(2\pi)^4}e^{-ip(x-y)} \\
&= i\,\delta^{(4)}(x-y).
\end{aligned}$$

光子は (9.58) から

$$(\partial^2 g_{\mu\nu} - i\epsilon g_{\mu\nu})_x D_F^{\nu\rho}(x-y) = (\partial^2 g_{\mu\nu} - i\epsilon g_{\mu\nu})_x \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2 + i\epsilon}g^{\nu\rho}e^{-ik(x-y)}$$

$$= i\delta_\mu{}^\rho\, \delta^{(4)}(x-y).$$

相互作用を含むラグランジアンは

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi^* - ieA_\mu\phi^*)(\partial_\mu\phi + ieA_\mu\phi) - m^2\phi^*\phi \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi^*)(\partial_\mu\phi) + ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - m^2\phi^*\phi + e^2A^2|\phi|^2 \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\partial^2 + m^2)\phi + ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2A^2|\phi|^2
\end{aligned}$$

（最後は部分積分を行い，表面項を無視した）．このうち，自由場は

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\partial^2 + m^2)\phi.$$

自由場の生成函数を次のように定義する：

$$\begin{aligned}
Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] &= \int\mathcal{D}A\int\mathcal{D}\phi\int\mathcal{D}\phi^*\exp\left[i\int d^4x\,(\mathcal{L}_0 + A_\mu J^\mu_{\text{em}} - J_s^*\phi + \phi^*J_s)\right] \\
&= \int\mathcal{D}\phi\int\mathcal{D}\phi^*\exp\left[i\int d^4x\,\left(\phi^*(-\partial^2 - m^2 + i\epsilon)\phi - J_s^*\phi + \phi^*J_s\right)\right] \\
&\quad\times\int\mathcal{D}A\exp\left[i\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu_{\text{em}}\right)\right].
\end{aligned}$$

第 1 項を計算する（x についての積分は省略）．(9.36) 以降の計算と同様に

$$\phi(x) = \phi'(x) + i\int d^4y\, D_F(x-y)J_s(y)$$

とおけば，

$$\begin{aligned}
&\phi^*(x)(-\partial^2 - m^2 + i\epsilon)\phi(x) - J_sx(x)^*\phi(x) + \phi^*(x)J_s(x) \\
&= \left[\phi'^*(x) - i\int d^4y\, D_F(x-y)J_s^*(y)\right](-\partial^2 - m^2 + i\epsilon)_x\left[\phi'(x) + i\int d^4y\, D_F(x-y)J_s(y)\right] \\
&\quad - J_s^*(x)\left[\phi'(x) + i\int d^4y\, D_F(x-y)J_s(y)\right] + J_s(x)\left[\phi'^*(x) - i\int d^4y\, D_F(x-y)J_s^*(y)\right] \\
&= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) - i\int d^4y\, D_F(x-y)J_s^*(y)(-\partial^2 - m^2 + i\epsilon)\phi'(x) \\
&\quad - \left[\phi'^*(x) - i\int d^4y\, D_F(x-y)J_s^*(y)\right]\int d^4y\,\delta^{(4)}(x-y)J_s(y) \\
&\quad - J_s^*(x)\phi'(x) + J_s(x)\phi'^*(x) - 2i\int d^4y\, J_s^*(x)D_F(x-y)J_s(y) \\
&= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) - i\int d^4y\,\left[(-\partial^2 - m^2 + i\epsilon)_x D_F(x-y)\right]J_s^*(y)\phi'(x) \\
&\quad - \left[\phi'^*(x) - i\int d^4y\, D_F(x-y)J_s^*(y)\right]J_s(x) \\
&\quad - J_s^*(x)\phi'(x) + J_s(x)\phi'^*(x) - 2i\int d^4y\, J_s^*(x)D_F(x-y)J_s(y) \\
&= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) + \int d^4y\,\delta^{(4)}(x-y)J_s^*(y)\phi'(x)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left[\phi'^*(x) - i\int d^4y\, D_F(x-y) J_s^*(y)\right] J_s(x) \\
&\quad - J_s^*(x)\phi'(x) + J_s(x)\phi'^*(x) - 2i\int d^4y\, J_s^*(x) D_F(x-y) J_s(y) \\
&= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) - i\int d^4y\, J_s^*(x) D_F(x-y) J_s(y).
\end{aligned}
$$

第 2 項を計算する（x についての積分は省略）.

$$
A_\mu(x) = A'_\mu(x) + i\int d^4y\, D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y)
$$

とおけば,

$$
\begin{aligned}
&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu_{\mathrm{em}} \\
&= \frac{1}{2}A_\mu(x)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A_\nu(x) + A_\mu(x) J^\mu_{\mathrm{em}}(x) \\
&= \frac{1}{2}\left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y)\right](\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})_x \left[A'_\nu(x) + i\int d^4y\, D^F_{\nu\rho}(x-y) J^\rho_{\mathrm{em}}(y)\right] \\
&\quad + \left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&= \frac{1}{2}A'_\mu(x)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A'_\nu(x) + \frac{i}{2}\int d^4y\, D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A'_\nu(x) \\
&\quad - \frac{1}{2}\left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&\quad + \left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&= \frac{1}{2}A'_\mu(x)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A'_\nu(x) + \frac{i}{2}\int d^4y\, (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})_x D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y) A'_\nu(x) \\
&\quad - \frac{1}{2}\left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&\quad + \left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&= \frac{1}{2}A'_\mu(x)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A'_\nu(x) - \frac{1}{2}\int d^4y\, \delta^\nu{}_\sigma\, \delta^{(4)}(x-y) J^\sigma_{\mathrm{em}}(y) A'_\nu(x) \\
&\quad - \frac{1}{2}\left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&\quad + \left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&= \frac{1}{2}A'_\mu(x)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A'_\nu(x) - \frac{1}{2}J^\nu_{\mathrm{em}}(x) A'_\nu(x) \\
&\quad - \frac{1}{2}\left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\sigma}(x-y) J^\sigma_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&\quad + \left[A'_\mu(x) + i\int d^4y\, D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y)\right] J^\mu_{\mathrm{em}}(x) \\
&= \frac{1}{2}A'_\mu(x)(\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})A'_\nu(x) + \frac{i}{2}\int d^4y\, J^\mu_{\mathrm{em}}(x) D^F_{\mu\nu}(x-y) J^\nu_{\mathrm{em}}(y).
\end{aligned}
$$

以上から,

$$
Z_{\mathrm{free}}[J_{\mathrm{em}}, J_s, J_s^*]
$$

$$= Z_{\text{free}}[0,0,0] \exp\left[\int d^4x \int d^4y \left(J_s^*(x) D_F(x-y) J_s(y) - \frac{1}{2} J_{\text{em}}^\mu(x) D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y)\right)\right]$$

を得る．

相互作用項

$$\mathcal{L}_{\text{int}} = ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2A^2|\phi|^2$$

を含めた生成函数は

$$\begin{aligned}
Z[J_{\text{em}}, J_s, J_s^*] &= Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp\left[i\int d^4x\, \mathcal{L}_{\text{int}}\right] \\
&= Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp\left[i\int d^4x\, e^2A^2|\phi|^2\right] \\
&\quad \times \exp\left[i\int d^4x\, ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)\right] \\
&\approx Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \left[1 + ie^2\int d^4x\, A(x)^2|\phi(x)|^2\right] \\
&\quad \times \left[1 - e\int d^4x\, A^\mu(x)\left\{\phi(x)\partial_\mu\phi^*(x) - \phi^*(x)\partial_\mu\phi(x)\right\}\right]
\end{aligned}$$

となる．

(9.35) と同様に汎函数微分をすれば伝播函数が求まる．例えば,

$$\langle 0|\, T\phi(x_1)\phi^*(x_2)\, |0\rangle = \frac{1}{Z_{\text{free}}[0,0,0]} \left(+i\frac{\delta}{\delta J_s^*(x_2)}\right)\left(-i\frac{\delta}{\delta J_s(x_1)}\right) Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*]\Big|_{J=0}.$$

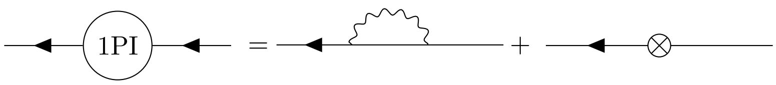
Chapter 10

Systematics of Renormalization

10.3 Renormalization of Quantum Electrodynamics

(10.42)

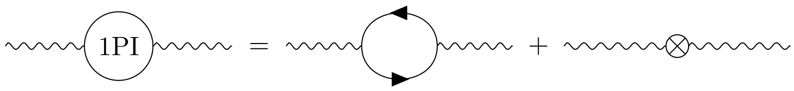
(10.39) で定義したように，



$$-i\Sigma(\not{p}) = -i\Sigma_2(\not{p}) + i(\not{p}\delta_2 - \delta_m).$$

(10.43)

(10.39) で定義したように，



$$i\Pi = i\Pi_2 - i\delta_3.$$

(10.45)

計算すべき函数は (6.38) に光子質量を導入し，次元正則化したもの：

$$\overline{u}(p')\delta\Gamma^\mu(p',p)u(p) = -ie^2 \int \frac{d^dk}{(2\pi)^d} \frac{\overline{u}(p')\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu u(p)}{[(k-p)^2-\mu^2+i\epsilon](k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}.$$

(6.43) と同様に分母を計算すれば

$$\ell = k + yq - zp, \quad \Delta = (1-z)^2m^2 + z\mu^2 - xyq^2$$

として

$$\frac{1}{[(k-p)^2-\mu^2+i\epsilon](k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} = \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\frac{2}{(\ell^2-\Delta)^3}.$$

分母を計算する．(A.55) を使えば

$$
\begin{aligned}
&\gamma^\nu(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu \\
&=\gamma^\nu\not{k}'\gamma^\mu\not{k}\gamma_\nu+m\gamma^\nu\not{k}'\gamma^\mu\gamma_\nu+m\gamma^\nu\gamma^\mu\not{k}\gamma_\nu+m^2\gamma^\nu\gamma^\mu\gamma_\nu \\
&=-2\not{k}\gamma^\mu\not{k}'+(4-d)\not{k}'\gamma^\mu\not{k}-(d-2)m^2\gamma^\mu+m[4(k+k')^\mu-(4-d)(\not{k}'\gamma^\mu+\gamma^\mu\not{k})].
\end{aligned}
\tag{10.3.1}
$$

簡単のため，

$$
k=\ell+a,\quad k'=\ell+b,\quad a=-yq+zp,\quad b=(1-y)q+zp
$$

とおく．ℓ^1 の項は積分すれば 0 なので無視し，[6.3.2] を使えば，

$$
\not{q}\gamma^\mu\not{q}=\not{q}(\gamma^\mu\gamma^\nu)q_\nu=\not{q}(2g^{\mu\nu}-\gamma^\nu\gamma^\mu)q_\nu=2q^\mu\not{q}-\not{q}\not{q}\gamma^\mu=-q^2\gamma^\mu.
$$

これらを使えば，

$$
\begin{aligned}
\not{k}\gamma^\mu\not{k}'&=(\not{\ell}+\not{a})\gamma^\mu(\not{\ell}+\not{b}) \\
&=\not{\ell}\gamma^\mu\not{\ell}+\not{a}\gamma^\mu\not{b} \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+(-y\not{q}+z\not{p})\gamma^\mu[(1-y)\not{q}+z\not{p}] \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[-y\not{q}+z(\not{p}'-\not{q})]\gamma^\mu[(1-y)\not{q}+zm] \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[-(y+z)\not{q}+z\not{p}']\gamma^\mu[(1-y)\not{q}+zm] \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[-(1-x)\not{q}+zm]\gamma^\mu[(1-y)\not{q}+zm] \\
&=\frac{2-d}{d}\ell^2\gamma^\mu-(1-x)(1-y)\not{q}\gamma^\mu\not{q}-(1-x)zm\not{q}\gamma^\mu+(1-y)zm\gamma^\mu\not{q}+z^2m^2\gamma^\mu \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+(1-x)(1-y)q^2\gamma^\mu-(1-x)zm\not{q}\gamma^\mu+(1-y)zm\gamma^\mu\not{q}+z^2m^2\gamma^\mu \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+(1-x)(1-y)q^2\gamma^\mu-(1-x)zm(2q^\mu-\gamma^\mu\not{q})+(1-y)zm\gamma^\mu\not{q}+z^2m^2\gamma^\mu \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+(1-x)(1-y)q^2\gamma^\mu-2(1-x)zmq^\mu+(2-x-y)zm\gamma^\mu\not{q}+z^2m^2\gamma^\mu \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+(1-x)(1-y)q^2\gamma^\mu-2(1-x)zmq^\mu+(1+z)zm\gamma^\mu\not{q}+z^2m^2\gamma^\mu
\end{aligned}
$$

および

$$
\begin{aligned}
\not{k}'\gamma^\mu\not{k}&=(\not{\ell}+\not{b})\gamma^\mu(\not{\ell}+\not{a}) \\
&=\not{\ell}\gamma^\mu\not{\ell}+\not{b}\gamma^\mu\not{a} \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[(1-y)\not{q}+z\not{p}]\gamma^\mu(-y\not{q}+z\not{p}) \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[(1-y)\not{q}+z(\not{p}'-\not{q})]\gamma^\mu(-y\not{q}+zm) \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[(1-y-z)\not{q}+z\not{p}']\gamma^\mu(-y\not{q}+zm) \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+[x\not{q}+zm]\gamma^\mu(-y\not{q}+zm) \\
&=\frac{2-d}{d}\ell^2\gamma^\mu-xy\not{q}\gamma^\mu\not{q}+xzm\not{q}\gamma^\mu-yzm\gamma^\mu\not{q}+z^2m^2\gamma^\mu \\
&=\frac{2-d}{d}\ell^2\gamma^\mu+xyq^2\gamma^\mu+xzm\not{q}\gamma^\mu-yzm\gamma^\mu\not{q}+z^2m^2\gamma^\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2-d}{d}\ell^2\gamma^\mu + xyq^2\gamma^\mu + xzm(2q^\mu - \gamma^\mu \not{q}) - yzm\gamma^\mu \not{q} + z^2m^2\gamma^\mu \\
&= \frac{2-d}{d}\ell^2\gamma^\mu + xyq^2\gamma^\mu + 2xzmq^\mu - (x+y)zm\gamma^\mu \not{q} + z^2m^2\gamma^\mu \\
&= \frac{2-d}{d}\ell^2\gamma^\mu + xyq^2\gamma^\mu + 2xzmq^\mu - (1-z)zm\gamma^\mu \not{q} + z^2m^2\gamma^\mu
\end{aligned}
$$

が得られる．したがって，[10.3.1] の第 1, 2 項は

$$
\begin{aligned}
&-2\not{k}\gamma^\mu \not{k}' + (4-d)\not{k}'\gamma^\mu \not{k} \\
&= -2\left[\frac{2-d}{d}\ell^2\gamma^\mu + (1-x)(1-y)q^2\gamma^\mu - 2(1-x)zmq^\mu + (1+z)zm\gamma^\mu \not{q} + z^2m^2\gamma^\mu\right] \\
&\quad + (4-d)\left[\frac{2-d}{d}\ell^2\gamma^\mu + xyq^2\gamma^\mu + 2xzmq^\mu - (1-z)zm\gamma^\mu \not{q} + z^2m^2\gamma^\mu\right] \\
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad - [2(1+z) + (4-d)(1-z)]\, zm\gamma^\mu \not{q} + [4(1-x) + 2(4-d)x]\, zmq^\mu + (2-d)z^2m^2\gamma^\mu \\
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad - [2(1+z) + (4-d)(1-z)]\, zm(2p'^\mu - 2m\gamma^\mu) + [4(1-x) + 2(4-d)x]\, zm(p'^\mu - p^\mu) \\
&\quad + (2-d)z^2m^2\gamma^\mu.
\end{aligned}
\tag{10.3.2}
$$

[10.3.1] の第 4 項は

$$
\begin{aligned}
4m(k^\mu + k'^\mu) &= 4m\left[(1-2y)q^\mu + 2zp^\mu\right] \\
&= 4m\left[(1-2y)(p'^\mu - p^\mu) + 2zp^\mu\right] \\
&= 4m\left[(1-2y)p'^\mu + (-1+2y+2z)p^\mu\right] \\
&= 4m\left[(1-2y)p'^\mu + (1-2x)p^\mu\right].
\end{aligned}
\tag{10.3.3}
$$

[10.3.1] の第 5 項は，[6.3.8] から

$$
\begin{aligned}
-(4-d)m(\not{k}'\gamma^\mu + \gamma^\mu \not{k}) &= -(4-d)m(2k^\mu + 2m\gamma^\mu - 2p^\mu) \\
&= -(4-d)m(-2yq^\mu + 2zp^\mu + 2m\gamma^\mu - 2p^\mu) \\
&= -(4-d)m[-2y(p'^\mu - p^\mu) + 2zp^\mu + 2m\gamma^\mu - 2p^\mu] \\
&= -(4-d)m[-2yp'^\mu + 2(y+z-1)p^\mu + 2m\gamma^\mu] \\
&= -(4-d)m[-2yp'^\mu - 2xp^\mu + 2m\gamma^\mu].
\end{aligned}
\tag{10.3.4}
$$

[10.3.2][10.3.3][10.3.4] から [10.3.1] は

$$
\begin{aligned}
&-2\not{k}\gamma^\mu \not{k}' + (4-d)\not{k}'\gamma^\mu \not{k} - (d-2)m^2\gamma^\mu + m[4(k+k')^\mu - (4-d)(\not{k}'\gamma^\mu + \gamma^\mu \not{k})] \\
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad - [2(1+z) + (4-d)(1-z)]\, zm(2p'^\mu - 2m\gamma^\mu) + [4(1-x) + 2(4-d)x]\, zm(p'^\mu - p^\mu) \\
&\quad + (2-d)z^2m^2\gamma^\mu - (d-2)m^2\gamma^\mu \\
&\quad + 4m\left[(1-2y)p'^\mu + (1-2x)p^\mu\right] - (4-d)m[-2yp'^\mu - 2xp^\mu + 2m\gamma^\mu]
\end{aligned}
$$

となり，p, p' を含まない項と含む項に分ければ

$$
\begin{aligned}
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad + 2\left[2(1+z) + (4-d)(1-z)\right] zm^2\gamma^\mu + (2-d)z^2m^2\gamma^\mu - (d-2)m^2\gamma^\mu - 2(4-d)m^2\gamma^\mu \\
&\quad - 2\left[2(1+z) + (4-d)(1-z)\right] zmp'^\mu + \left[4(1-x) + 2(4-d)x\right] zm(p'^\mu - p^\mu) \\
&\quad + 4m\left[(1-2y)p'^\mu + (1-2x)p^\mu\right] + 2(4-d)m[yp'^\mu + xp^\mu].
\end{aligned} \qquad [10.3.5]
$$

p, p' を含まない項のうち，m を含む項は $m^2\gamma^\mu\times$

$$
\begin{aligned}
&2\left[2(1+z) + (4-d)(1-z)\right] z + (2-d)z^2 - (d-2) - 2(4-d) \\
&= 4(1+z)z + 2(4-d)(1-z)z + (4-d)z^2 - 2z^2 + (4-d) - 2 - 2(4-d) \\
&= 2(z^2 + 2z - 1) - (4-d)(z-1)^2.
\end{aligned}
$$

p を含む項は $mp^\mu\times$

$$
\begin{aligned}
-4(1-x)z - 2(4-d)xz + 4(1-2x) + 2(4-d)x &= 4[1 - 2x - z + xz] + 2(4-d)x(1-z) \\
&= 4[1 - 2y - z + yz] + 2(4-d)y(1-z).
\end{aligned}
$$

p' を含む項は $mp'^\mu\times$

$$
\begin{aligned}
&-4(1+z)z - 2(4-d)(1-z)z + 4(1-x)z + 2(4-d)xz + 4(1-2y) + 2(4-d)y \\
&= 4[1 - 2y - xz - z^2] + 2(4-d)[xz + y - z + z^2] \\
&= 4[1 - 2y - xz - z(1-x-y)] + 2(4-d)[xz + y - z + z(1-x-y)] \\
&= 4[1 - 2y - z + yz] + 2(4-d)y(1-z).
\end{aligned}
$$

さらに，

$$
\begin{aligned}
\int_0^1 dz \int_0^{1-z} dy\,(1 - 2y - z + yz) &= \int_0^1 dz\,\left[(1-z) - (1-z)^2 - (1-z)z + \frac{1}{2}(1-z)^2 z\right] \\
&= \int_0^1 dz \int_0^{1-z} dy\,\frac{1}{2}z(1-z)
\end{aligned}
$$

及び

$$
\int_0^1 dz \int_0^{1-z} dy\,y(1-z) = \int_0^1 dz\,\frac{1}{2}(1-z)^3 = \int_0^1 dz \int_0^{1-z} dy\,\frac{1}{2}(1-z)^2
$$

なので，p, p' を含む項は

$$
[4(1 - 2y - z + yz) + 2(4-d)y(1-z)]\,m(p^\mu + p'^\mu) = \left[2z(1-z) + (4-d)(1-z)^2\right] m(p^\mu + p'^\mu).
$$

以上から，

$$
\begin{aligned}
[10.3.5] &= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad + \left[2(z^2 + 2z - 1) - (4-d)(z-1)^2\right] m^2\gamma^\mu \\
&\quad + \left[2z(1-z) + (4-d)(1-z)^2\right] m(p^\mu + p'^\mu) \\
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad + \left[2(z^2 + 2z - 1) - (4-d)(z-1)^2\right] m^2\gamma^\mu
\end{aligned}
$$

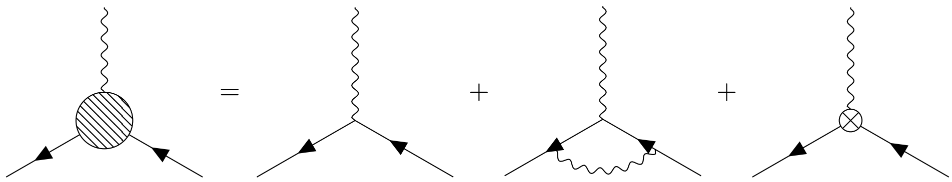
$$
\begin{aligned}
&+ \left[2z(1-z) + (4-d)(1-z)^2\right] m(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) \\
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu - 2(1-x)(1-y)q^2\gamma^\mu + (4-d)xyq^2\gamma^\mu \\
&\quad + \left[2(-z^2+4z-1) + (4-d)(z-1)^2\right] m^2\gamma^\mu \\
&\quad - \left[2z(1-z) + (4-d)(1-z)^2\right] im\sigma^{\mu\nu}q_\nu \\
&= \frac{(2-d)^2}{d}\ell^2\gamma^\mu \\
&\quad - \left(q^2[2(1-x)(1-y) - \epsilon xy] + m^2[2(1-4z+z^2) - \epsilon(1-z)^2]\right)\gamma^\mu \\
&\quad - \left[2z(1-z) + (4-d)(1-z)^2\right] im\sigma^{\mu\nu}q_\nu .
\end{aligned}
$$

(6.33) より，初め 2 項が F_1 への寄与．したがって，

$$
\begin{aligned}
\delta F_1(q^2) &= -2ie^2 \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}\frac{(2-d)^2}{d}\ell^2 \\
&\quad + 2ie^2 \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}\left(q^2[\cdots] + m^2[\cdots]\right) \\
&= \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx\,dy\,dz\,\delta(x+y+z-1) \Bigg[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\frac{(2-\epsilon)^2}{2} \\
&\quad + \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}\left(q^2[2(1-x)(1-y) - \epsilon xy] + m^2[2(1-4z+z^2) - \epsilon(1-z)^2]\right)\Bigg].
\end{aligned}
$$

(10.46)

(10.39) で定義したように，



$$
-ie\Gamma^\mu(p',p) = -ie\gamma^\mu - ie\left[\gamma^\mu\delta F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right] - ie\gamma^\mu\delta_1.
$$

Problems

Problem 10.1: One-loop structure of QED

ガンマ行列 8 個の積を計算する．

$$
\begin{aligned}
\mathrm{Tr}[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}] &= \mathrm{Tr}[\gamma^\mu\not{k}(2k^\nu - \not{k}\gamma^\nu)\gamma^\rho\not{k}(2k^\sigma - \not{k}\gamma^\sigma)] \\
&= 4k^\nu k^\sigma\,\mathrm{Tr}[\gamma^\mu\not{k}\gamma^\rho\not{k}] - 2k^\nu\,\mathrm{Tr}[\gamma^\mu\not{k}\gamma^\rho\not{k}\not{k}\gamma^\sigma] \\
&\quad - 2k^\sigma\,\mathrm{Tr}[\gamma^\mu\not{k}\not{k}\gamma^\nu\gamma^\rho\not{k}] + \mathrm{Tr}[\gamma^\mu\not{k}\not{k}\gamma^\nu\gamma^\rho\not{k}\not{k}\gamma^\sigma] \\
&= 4k^\nu k^\sigma\,\mathrm{Tr}[\gamma^\mu\not{k}\gamma^\rho\not{k}] - 2k^2k^\nu\,\mathrm{Tr}[\gamma^\mu\not{k}\gamma^\rho\gamma^\sigma] \\
&\quad - 2k^2k^\sigma\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\not{k}] + k^4\,\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]
\end{aligned}
$$

$$
\begin{aligned}
&= 16k^\nu k^\sigma(2k^\mu k^\rho - k^2 g^{\mu\rho}) - 8k^2 k^\nu(k^\mu g^{\rho\sigma} - g^{\mu\rho}k^\sigma + g^{\mu\sigma}k^\rho) \\
&\quad - 8k^2 k^\sigma(g^{\mu\nu}k^\rho - g^{\mu\rho}k^\nu + k^\mu g^{\nu\rho}) + 4k^4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&= 32k^\mu k^\nu k^\rho k^\sigma - 8k^2(k^\mu k^\nu g^{\rho\sigma} + k^\rho k^\sigma g^{\mu\nu} + k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\rho g^{\mu\sigma}) \\
&\quad + 4k^4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).
\end{aligned}
$$

(A.41)(A.42) から

$$
\begin{aligned}
&\to 3k^4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) - 2k^4(g^{\mu\nu}g^{\rho\sigma} + g^{\rho\sigma}g^{\mu\nu} + g^{\mu\sigma}g^{\nu\rho} + g^{\nu\rho}g^{\mu\sigma}) \\
&\quad + 4k^4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&= 3k^4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) - 4k^4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 4k^4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&= 3k^4(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}).
\end{aligned}
$$

Chapter 11

Renormalization and Symmetry

11.4 Computation of Effective Action

(11.67)

汎函数微分について．Euler-Lagrange 方程式を導く際の変分と同様にして

$$
\begin{aligned}
\frac{\delta\mathcal{L}[\phi,\ldots]}{\delta\phi} &= \frac{\mathcal{L}[\phi+\delta\phi,\ldots]-\mathcal{L}[\phi,\ldots]}{\delta\phi} \\
&= \frac{1}{\delta\phi}\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)+\cdots\right]\delta\phi \\
&= \frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right).
\end{aligned}
$$

$(\partial_\mu\phi^i_{\mathrm{cl}})^2/2$ を 2 階変分すれば，$-\partial^2\delta^{ij}$ が得られる．

Chapter 12

The Renormalization Group

12.1 Wilson's Approach to Renormalization Theory

(12.8)

汎関数積分のために (12.7) を和の形に書いておく：

$$\int \mathcal{L}_0 = \frac{1}{2} \sum_{|k|=b\Lambda}^{\Lambda} \frac{k^2}{(2\pi)^d} |\hat{\phi}(k)|^2.$$

(9.23) と同様にして

$$\int \mathcal{D}\hat{\phi}\, \exp\left(-\int \mathcal{L}_0\right) = \prod_{b\Lambda \le |k| < \Lambda} \sqrt{(2\pi)^d \frac{\pi}{k^2}}.$$

$b\Lambda \le |k| < \Lambda$ でない場合は $\hat{\phi}(k) = 0$．以下では $b\Lambda \le |k| < \Lambda$ の場合を考える．(9.26) と同様にして

$$\begin{aligned}
&\int \mathcal{D}\hat{\phi}\, \exp\left(-\int \mathcal{L}_0\right) \hat{\phi}(k)\hat{\phi}(p) \\
&= \left(\prod_{b\Lambda \le |l| < \Lambda} \int d\,\mathrm{Re}\,\hat{\phi}(l)\, d\,\mathrm{Im}\,\hat{\phi}(l) \right) \Big(\mathrm{Re}\,\hat{\phi}(k) + i\,\mathrm{Im}\,\hat{\phi}(k)\Big) \Big(\mathrm{Re}\,\hat{\phi}(p) + i\,\mathrm{Im}\,\hat{\phi}(p)\Big) \\
&\quad \times \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\mathrm{Re}\,\hat{\phi}(l))^2 \right] \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\mathrm{Im}\,\hat{\phi}(l))^2 \right].
\end{aligned}$$

(9.26) の後で説明されているように，積分が非零となるのは $k + p = 0$ の場合のみ．

$$\begin{aligned}
&\int \mathcal{D}\hat{\phi}\, \exp\left(-\int \mathcal{L}_0\right) \hat{\phi}(k)\hat{\phi}(-k) \\
&= \left(\prod_{b\Lambda \le |l| < \Lambda} \int d\,\mathrm{Re}\,\hat{\phi}(l)\, d\,\mathrm{Im}\,\hat{\phi}(l) \right) \Big[(\mathrm{Re}\,\hat{\phi}(k))^2 + (\mathrm{Im}\,\hat{\phi}(k))^2\Big] \\
&\quad \times \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\mathrm{Re}\,\hat{\phi}(l))^2 \right] \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\mathrm{Im}\,\hat{\phi}(l))^2 \right] \\
&= 2 \int d\,\mathrm{Re}\,\hat{\phi}(k)\, (\mathrm{Re}\,\hat{\phi}(k))^2 \exp\left[-\frac{1}{2} \frac{k^2}{(2\pi)^d} (\mathrm{Re}\,\hat{\phi}(k))^2 \right] \int d\,\mathrm{Im}\,\hat{\phi}(k)\, \exp\left[-\frac{1}{2} \frac{k^2}{(2\pi)^d} (\mathrm{Im}\,\hat{\phi}(k))^2 \right]
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{\substack{b\Lambda\le|l|<\Lambda \\ l\neq k}} \int d\,\mathrm{Re}\,\hat{\phi}(l)\, \exp\left[-\frac{1}{2}\frac{l^2}{(2\pi)^d}(\mathrm{Re}\,\hat{\phi}(l))^2\right] \int d\,\mathrm{Im}\,\hat{\phi}(l)\, \exp\left[-\frac{1}{2}\frac{l^2}{(2\pi)^d}(\mathrm{Im}\,\hat{\phi}(l))^2\right] \\
&= \frac{(2\pi)^d}{k^2} \prod_{b\Lambda\le|k|<\Lambda} \sqrt{(2\pi)^d \frac{\pi}{k^2}}.
\end{aligned}
$$

以上から

$$
\overbracket{\hat{\phi}(k)\hat{\phi}}(p) = \frac{(2\pi)^d}{k^2}\,\delta^{(d)}(k+p)\Theta(k).
$$

(12.10)

(12.8) から

$$
\begin{aligned}
\overbracket{\hat{\phi}(x)\hat{\phi}}(x) &= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{-i(k+p)\cdot x}\,\overbracket{\hat{\phi}(k)\hat{\phi}}(p) \\
&= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{-i(k+p)\cdot x}\,\frac{(2\pi)^d}{k^2}\,\delta^{(d)}(k+p)\Theta(k) \\
&= \int_{b\Lambda\le|k|<\Lambda} \frac{d^dk}{(2\pi)^d}\frac{1}{k^2}.
\end{aligned}
$$

よって，

$$
\begin{aligned}
-\int d^dx\,\frac{\lambda}{4}\phi(x)\phi(x)\overbracket{\hat{\phi}(x)\hat{\phi}}(x) &= -\int d^dx\,\frac{\lambda}{4}\phi(x)\phi(x)\int_{b\Lambda\le|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} \\
&= -\frac{\mu}{2}\int d^dx\,\phi(x)\phi(x) \\
&= -\frac{\mu}{2}\int d^dx \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{-i(k+p)\cdot x}\phi(k)\phi(p) \\
&= -\frac{\mu}{2}\int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\,\delta^{(d)}(k+p)\phi(k)\phi(p) \\
&= -\frac{\mu}{2}\int \frac{d^dk}{(2\pi)^d}\phi(k)\phi(-k).
\end{aligned}
$$

(12.14)

(12.8) から

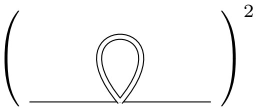
$$
\begin{aligned}
\overbracket{\hat{\phi}(x)\hat{\phi}}(y) &= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{-i(k\cdot x+p\cdot y)}\,\overbracket{\hat{\phi}(k)\hat{\phi}}(p) \\
&= \int \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{-i(k\cdot x+p\cdot y)}\,\frac{(2\pi)^d}{k^2}\,\delta^{(d)}(k+p)\Theta(k) \\
&= \int_{b\Lambda\le|k|<\Lambda} \frac{d^dk}{(2\pi)^d} e^{-ik\cdot(x-y)}\,\frac{1}{k^2}.
\end{aligned}
$$

$\exp(-\lambda\phi^2\hat{\phi}^2/4)$ の 2 次の展開

$$
\frac{1}{2}\int d^dx\,\frac{\lambda}{4}\phi(x)\phi(x)\hat{\phi}(x)\hat{\phi}(x)\int d^dy\,\frac{\lambda}{4}\phi(y)\phi(y)\hat{\phi}(y)\hat{\phi}(y)
$$

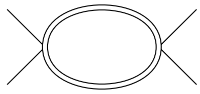
を考える．$\hat{\phi}$ の縮約には

$$\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y) =$$



および

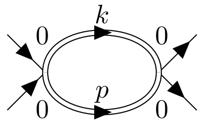
$$\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y) =$$



の 2 通りがある．2 つ目の縮約は 2 通りあるので，

$$\begin{aligned}
&\frac{\lambda^2}{16}\int d^dx\, d^dy\, \phi(x)\phi(x)\phi(y)\phi(y)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y) \\
&= \frac{\lambda^2}{16}\int d^dx\, d^dy\, \phi^2(x)\phi^2(y) \int\limits_{\substack{b\Lambda\le|k|<\Lambda \\ b\Lambda\le|p|<\Lambda}} \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d} e^{-i(k+p)\cdot(x-y)}\frac{1}{k^2}\frac{1}{p^2} \\
&= \frac{\lambda^2}{16}\int\limits_{\substack{b\Lambda\le|k|<\Lambda \\ b\Lambda\le|p|<\Lambda}} \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{1}{k^2}\frac{1}{p^2}\int d^dx\, \phi^2(x)e^{-i(k+p)\cdot x}\int d^dy\, \phi^2(y)e^{i(k+p)\cdot y} \\
&= \frac{\lambda^2}{16}\int\limits_{\substack{b\Lambda\le|k|<\Lambda \\ b\Lambda\le|p|<\Lambda}} \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{1}{k^2}\frac{1}{p^2}\mathcal{F}[\phi^2](-k-p)\mathcal{F}[\phi^2](k+p) \\
&= \frac{\lambda^2}{16}\int\limits_{\substack{b\Lambda\le|k|<\Lambda \\ b\Lambda\le|p|<\Lambda}} \frac{d^dk}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{1}{k^2}\frac{1}{p^2}\left|\mathcal{F}[\phi^2](k+p)\right|^2.
\end{aligned}$$

ϕ の運動量に関する条件から，$\mathcal{F}[\phi^2](k+p) \approx \mathcal{F}[\phi^2](0)\,\delta^{(d)}(k+p)$.



よって，

$$\begin{aligned}
&\approx \frac{1}{(2\pi)^d}\frac{\lambda^2}{16}\int\limits_{b\Lambda\le|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\left|\mathcal{F}[\phi^2](0)\right|^2 \\
&\approx \frac{\lambda^2}{16}\int\limits_{b\Lambda\le|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\int_{-\infty}^{\infty}\frac{d^dp}{(2\pi)^d}\left|\mathcal{F}[\phi^2](p)\right|^2 \\
&= \frac{\lambda^2}{16}\int\limits_{b\Lambda\le|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\left\|\mathcal{F}[\phi^2]\right\|^2 \\
&= \frac{\lambda^2}{16}\int\limits_{b\Lambda\le|k|<\Lambda}\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\left\|\phi^2\right\|^2
\end{aligned}$$

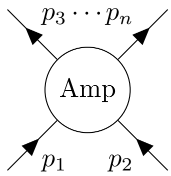
$$
\begin{aligned}
&= \frac{\lambda^2}{16} \int\limits_{b\Lambda \le |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^2 \int d^d x\, \phi^4(x) \\
&= -\frac{1}{4!} \int d^d x\, \zeta \phi^4.
\end{aligned}
$$

12.2 The Callan-Symanzik Equation

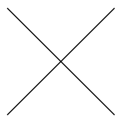
(12.52)

n 頂点の Green 函数を考える．(7.42)(12.35) から

$$
\begin{aligned}
\langle \Omega | T\{\phi(p_1)\phi(p_2)\cdots\phi(p_n)\} |\Omega\rangle &= Z^{-n/2} \langle \Omega | T\{\phi_0(p_1)\phi_0(p_2)\cdots\phi(p_n)\} |\Omega\rangle \\
&= Z^{-n/2} \prod_{i=1}^{n} \frac{\sqrt{Z} i}{{p_i}^2} \langle \boldsymbol{p} | S |\boldsymbol{p}\rangle \\
&= \left(\prod_{i=1}^{n} \frac{i}{{p_i}^2}\right)
\end{aligned}
$$

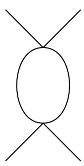


となる[*1]．tree-level は，



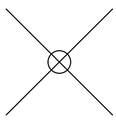
$$= -ig.$$

1PI-loop は，(10.20) などで計算したように，$\log(-p^2)$ の発散を持つ：



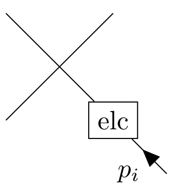
$$+ \cdots = -iB \log \frac{\Lambda^2}{-p^2}.$$

vertex counterterm は，

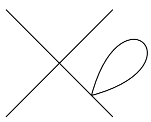


$$= -i\delta_g.$$

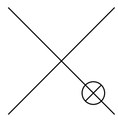
運動量 p_i の外線に対する external leg correction は



$$=$$



$$+$$



$$= (-ig) A_i \log \frac{\Lambda^2}{-p^2} + (-ig) \times i{p_i}^2 \delta_{Z_i} \frac{i}{{p_i}^2}$$

[*1] くりこみした量でダイアグラムを計算するので，(7.45) の右辺の $\sqrt{Z}$ は不要

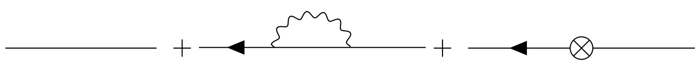
$$= (-ig)\left(A_i \log \frac{\Lambda^2}{-p^2} - \delta_{Z_i}\right)$$

となる（g^1 の項のみ考えるので，i について和を取れば良い）.

(12.57)

$G^{(2,0)}(p)$ を求める．電子の自己エネルギー (7.15) を使えば (12.49) と同様に

$G^{(2,0)}(p) =$



$$\begin{aligned}
&= \frac{i}{\not p} + \frac{i}{\not p}[-i\Sigma_2]\frac{i}{\not p} + \frac{i}{\not p}i(\not p\delta_2)\frac{i}{\not p} \\
&= \frac{i}{\not p} + \frac{i}{\not p}[-i\Sigma_2]\frac{i}{\not p} - \frac{i}{\not p}\delta_2.
\end{aligned}$$

(12.50) と同様に，Callan-Syamzik 方程式から

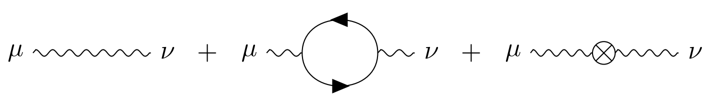
$$-\frac{i}{\not p}M\frac{\partial}{\partial M}\delta_2 + 2\gamma_2\frac{i}{\not p} = 0$$

なので

$$\gamma_2 = \frac{1}{2}M\frac{\partial}{\partial M}\delta_2.$$

$G^{(0,2)}(q)$ を求める．真空偏極 (7.71)(7.73) を使えば (12.49) と同様に

$G^{(0,2)}(q) =$



$$\begin{aligned}
&= \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \\
&\quad + \frac{-i}{q^2}\left(\delta^\mu{}_\rho - \frac{q^\mu q_\rho}{q^2}\right) i(g^{\rho\sigma}q^2 - q^\rho q^\sigma)\Pi_2 \frac{-i}{q^2}\left(\delta^\nu{}_\sigma - \frac{q^\nu q_\sigma}{q^2}\right) \\
&\quad + \frac{-i}{q^2}\left(\delta^\mu{}_\rho - \frac{q^\mu q_\rho}{q^2}\right) (-i)(g^{\rho\sigma}q^2 - q^\rho q^\sigma)\delta_3 \frac{-i}{q^2}\left(\delta^\nu{}_\sigma - \frac{q^\nu q_\sigma}{q^2}\right) \\
&= \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \\
&\quad + \frac{-i}{q^2}(g^{\mu\nu}q^2 - q^\mu q^\nu)i\Pi_2\frac{-i}{q^2} + \frac{-i}{q^2}(g^{\mu\nu}q^2 - q^\mu q^\nu)(-i)\delta_3\frac{-i}{q^2} \\
&= \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)[1 + \Pi_2 - \delta_3].
\end{aligned}$$

(12.50) と同様に，Callan-Syamzik 方程式から

$$-M\frac{\partial}{\partial M}\delta_3 + 2\gamma_3 = 0$$

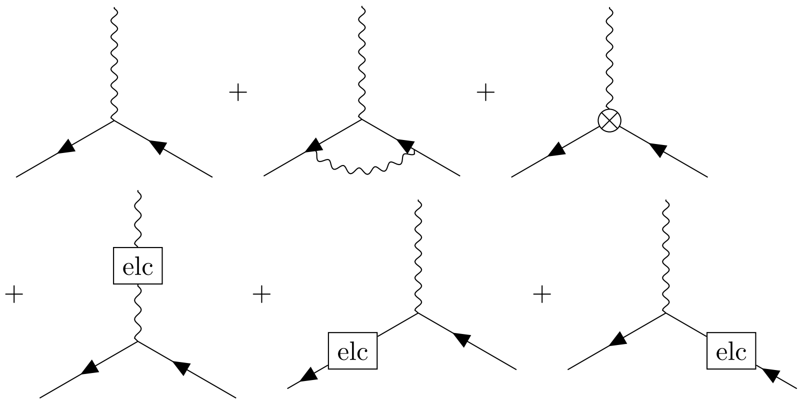
なので

$$\gamma_3 = \frac{1}{2}M\frac{\partial}{\partial M}\delta_3.$$

(12.58)

$G^{(2,1)}(p_1, p_2, q)$ を求める．頂点補正 (6.38) を使えば (12.52) と同様に

$G^{(2,1)}(p_1, p_2, q) =$



$$
\begin{aligned}
&= \frac{i}{\not{p}_1}(-ie\gamma^\mu)\frac{i}{\not{p}_2}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \frac{i}{\not{p}_1}(-ie\delta\Gamma^\mu)\frac{i}{\not{p}_2}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \\
&\quad + \frac{i}{\not{p}_1}(-ie\gamma^\mu\delta_1)\frac{i}{\not{p}_2}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \\
&\quad + \frac{i}{\not{p}_1}(-ie\gamma^\mu)(\Pi_2 - \delta_3)\frac{i}{\not{p}_2}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \\
&\quad + \frac{i}{\not{p}_1}\left(1 + \Sigma_2\frac{1}{\not{p}_1} - \delta_2\right)(-ie\gamma^\mu)\frac{i}{\not{p}_2}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \\
&\quad + \frac{i}{\not{p}_1}(-ie\gamma^\mu)\left(1 + \Sigma_2\frac{1}{\not{p}_2} - \delta_2\right)\frac{i}{\not{p}_2}\frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right).
\end{aligned}
$$

(12.53) と同様に，Callan-Syamzik 方程式から

$$M\frac{\partial}{\partial M}(\delta_1 - 2\delta_2 - \delta_3) + \frac{\beta}{e} + 2\gamma_2 + \gamma_3 = 0.$$

(12.57) を代入して

$$\beta = eM\frac{\partial}{\partial M}\left(-\delta_1 + \delta_2 + \frac{\delta_3}{2}\right).$$

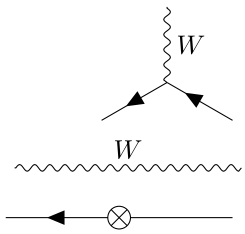
12.4 Renormalization of Local Operators

(12.112)

相互作用項は

$$\delta\mathcal{L} = \frac{g}{\sqrt{2}}W_\mu\bar{\psi}\gamma^\mu(1-\gamma^5)\psi$$

なので



$$
\begin{aligned}
&= \frac{ig}{\sqrt{2}}\gamma^\mu(1-\gamma^5), \\
&= \frac{-ig^{\mu\nu}}{q^2 - {m_W}^2 + i\epsilon}, \\
&= i\not{p}\delta_Z.
\end{aligned}
$$

(12.57) で γ_2 を導出したのと同様に

$$
\gamma = \frac{1}{2} M \frac{\partial}{\partial M}\delta_Z.
$$

$G^{(m,1)}$ は

$$
\begin{aligned}
&\text{(tree level)} + \text{(1PI loop)} + \text{(external leg corrections)} + \text{(operator counterterm)} \\
&= \text{(TL)} + \text{(TL)} \times B + \text{(TL)} \times m(A - \delta_Z) + \text{(TL)} \times \delta_{\mathcal{O}}
\end{aligned}
$$

と表せる（A はフェルミオンの自己エネルギー，$-\delta_Z$ はフェルミオンの相殺項 $i\not{p}\delta_Z$ と伝播函数 $i/\not{p}$ の積）ので，Callan-Symanzik 方程式から

$$
M\frac{\partial}{\partial M}(\delta_{\mathcal{O}} - m\delta_Z) + m\gamma + \gamma_{\mathcal{O}} = 0.
$$

よって，

$$
\gamma_{\mathcal{O}} = M\frac{\partial}{\partial M}\left(-\delta_{\mathcal{O}} + \frac{m}{2}\delta_Z\right).
$$

12.5 Evolution of Mass Parameters

(12.123)

$C \to \rho M^{4-d}$ と変換する．変換前の Callan-Symanzik 方程式は

$$
\left[M\frac{\partial}{\partial M} + \gamma C \frac{\partial}{\partial C} + \cdots\right] G(M, C) = 0.
$$

変換後は

$$
\begin{aligned}
M\frac{\partial}{\partial M} G(M, \rho M^{4-d}) &= M\frac{\partial}{\partial M}G(M,C) + M\frac{\partial \rho M^{4-d}}{\partial M}\frac{\partial}{\partial \rho M^{4-d}} G(M, \rho M^{4-d}) \\
&= M\frac{\partial}{\partial M}G(M,C) + (4-d)\frac{\partial}{\partial \rho}G(M, \rho M^{4-d}).
\end{aligned}
$$

及び

$$
\gamma C \frac{\partial}{\partial C} G(M,C) = \gamma\rho\frac{\partial}{\partial\rho}G(M,\rho M^{4-d})
$$

となるので，Callan-Symanzik 方程式は

$$
\left[M\frac{\partial}{\partial M} + (\gamma + d - 4)\rho\frac{\partial}{\partial\rho} + \cdots\right] G(M, \rho M^{4-d}) = 0
$$

と修正される．

(12.131)

$d=4$ の Callan-Symanzik 方程式は既知なので，d 次元を考える際は，これに帰着させれば良い．d 次元の場合は $\mathcal{L}$ を (12.129) の様に変形すれば，形式的に $d=4$ となる．すなわち，$d=4$ のラグランジアン $\mathcal{L}^{(4)}$ の各項に適当な M の冪乗をかけることによって，d 次元の場合を表すことができる．

質量項の補正

d 次元のラグランジアンの質量項は

$$\frac{1}{2}\rho_m M^2 {\phi_M}^2$$

である（$d=4$ の場合と同じ）．$d=4$ の Callan-Symanzik 方程式 (12.119)

$$\left[M\frac{\partial}{\partial M}+\gamma_{\phi^2}^{(4)} m^2\frac{\partial}{\partial m^2}+\cdots\right]G(M,m^2)=0$$

で $m^2\to\rho_m M^2$ とすれば d 次元の方程式が得られる．(12.123) と同様に

$$M\frac{\partial}{\partial M}G(M,\rho_m M^2)=M\frac{\partial}{\partial M}G(M,m^2)+2\rho_m\frac{\partial}{\partial\rho_m}G(M,\rho_m M^2)$$

及び

$$\gamma_{\phi^2}^{(4)} m^2\frac{\partial}{\partial m^2}G(M,m^2)=\gamma_{\phi^2}^{(4)}\rho_m\frac{\partial}{\partial\rho_m}G(M,\rho_m M^2)$$

なので，Callan-Symanzik 方程式は

$$\left[M\frac{\partial}{\partial M}+(\gamma_{\phi^2}^{(4)}-2)\rho_m\frac{\partial}{\partial\rho_m}+\cdots\right]G(M,\rho_m M^2)=0$$

と修正される．すなわち質量のベータ函数は

$$\beta_m=(\gamma_{\phi^2}^{(4)}-2)\rho_m.$$

相互作用項の補正

$d=4$ での相互作用項を

$$\frac{\lambda}{4}{\phi_M}^4\to\frac{\lambda}{4}M^{4-d}{\phi_M}^4$$

とすれば d 次元の相互作用項を表すことができる．$d=4$ の Callan-Symanzik 方程式 (12.119)

$$\left[M\frac{\partial}{\partial M}+\beta^{(4)}\frac{\partial}{\partial\lambda}+\cdots\right]G(M,\lambda)=0$$

で $\lambda\to M^{4-d}\lambda$ とすれば d 次元の方程式が得られる．(12.123) と同様に

$$M\frac{\partial}{\partial M}G(M,M^{4-d}\lambda)=M\frac{\partial}{\partial M}G(M,\lambda)+(4-d)\lambda\frac{\partial}{\partial\lambda}G(M,M^{4-d}\lambda)$$

及び

$$\beta^{(4)}\frac{\partial}{\partial\lambda}G(M,\lambda)=\beta^{(4)}\frac{\partial}{\partial\lambda}G(M,M^{4-d}\lambda)$$

なので，Callan-Symanzik 方程式は

$$\left[M\frac{\partial}{\partial M}+[(d-4)\lambda+\beta^{(4)}]\frac{\partial}{\partial\lambda}+\cdots\right]G(M,M^{4-d}\lambda)=0$$

と修正される．すなわち相互作用のベータ函数は

$$\beta=(d-4)\lambda+\beta^{(4)}.$$

一般作用素の補正

$d=4$ での一般作用素を

$$\rho_i M^{4-d_i}\mathcal{O}^i_M(x)\to\rho_i M^{d-d_i}\mathcal{O}^i_M(x)$$

とすれば d 次元の相互作用項を表すことができる．$d=4$ の Callan-Symanzik 方程式 (12.123)

$$\left[M\frac{\partial}{\partial M}+(d_i-4+\gamma_i^{(4)})\rho_i\frac{\partial}{\partial\rho_i}+\cdots\right]G(M,\rho_i M^{4-d})=0$$

で $M^{4-d_i}\to M^{d-d_i}$ とすれば d 次元の方程式が得られる．(12.123) と同様に

$$M\frac{\partial}{\partial M}G(M,\rho_i M^{d-d_i})=M\frac{\partial}{\partial M}G(M,\rho_i M^{4-d_i})+(d-4)\rho_i\frac{\partial}{\partial\rho_i}G(M,\rho_i M^{d-d_i})$$

及び

$$(d_i-4+\gamma_i^{(4)})\rho_i\frac{\partial}{\partial\rho_i}G(M,\rho_i M^{4-d})=(d_i-4+\gamma_i^{(4)})\rho_i\frac{\partial}{\partial\rho_i}G(M,\rho_i M^{d-d_i})$$

なので，Callan-Symanzik 方程式は

$$\left[M\frac{\partial}{\partial M}+(d_i-d+\gamma_i^{(4)})\rho_i\frac{\partial}{\partial\rho_i}+\cdots\right]G(M,M^{d-d_i}\rho_i)=0$$

と修正される．すなわち一般作用素のベータ函数は

$$\beta_i=(d_i-d+\gamma_i^{(4)})\rho_i.$$

Problems

Problem 12.2: Beta function of the Gross-Neveu model

Gross-Neveu 模型は

$$\mathcal{L}=\sum_{i=1}^N\bar\psi_i(i\partial\!\!\!/)\psi_i+\frac{g^2}{2}\left(\sum_{i=1}^N\bar\psi_i\psi_i\right)$$

で与えられる．$d=2$ の Dirac 行列は

$$(\gamma^0)_{\alpha\beta}=\begin{pmatrix} & -i\\ i & \end{pmatrix},\quad(\gamma^1)_{\alpha\beta}=\begin{pmatrix} & i\\ i & \end{pmatrix}$$

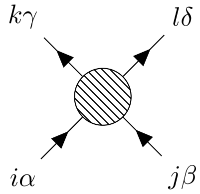
である（スピノルの添字を $\alpha=0,1$ などのギリシャ文字で表す）．

伝播函数は

$$i\alpha \xrightarrow{\ p\ } j\beta \quad = \left(\frac{i}{\not p}\right)_{\beta\alpha} \delta_{ij}$$

で与えられる．ガンマ行列は対角成分を持たないので，$\alpha = \beta$ なら伝播函数は 0 である．

4 点相関函数



$$= \langle\Omega| \, T\{\psi_{i\alpha}(x_1)\psi_{j\beta}(x_2)\bar\psi_{k\gamma}(x_3)\bar\psi_{l\delta}(x_4)\} \, |\Omega\rangle$$

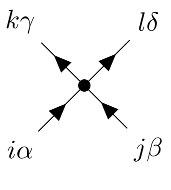
を求める．(4.31) から，1 次の展開は

$$\langle 0| \, T\{\psi_{i\alpha}(x_1)\psi_{j\beta}(x_2)\bar\psi_{k\gamma}(x_3)\bar\psi_{l\delta}(x_4) \left(i\frac{g^2}{2}\right)\int d^4x \sum_{mn}\sum_{\rho\sigma} \bar\psi_{m\rho}(x)\psi_{m\rho}(x)\bar\psi_{n\sigma}(x)\psi_{n\sigma}(x)\} \, |0\rangle$$

となる．縮約の方法には 4 通りある：

1. $\psi_{i\alpha}\bar\psi_{k\gamma}\bar\psi_{m\rho}\psi_{m\rho}$ なら
$$m=i,\quad m=k,\quad \rho\neq\alpha,\quad \rho\neq\gamma;\quad n=j,\quad n=l,\quad \sigma\neq\beta,\quad \sigma\neq\delta \quad \therefore \quad \delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}.$$
2. $\psi_{j\beta}\bar\psi_{l\delta}\bar\psi_{m\rho}\psi_{m\rho}$ なら
$$m=j,\quad m=l,\quad \rho\neq\beta,\quad \rho\neq\delta;\quad n=i,\quad n=k,\quad \sigma\neq\alpha,\quad \sigma\neq\gamma \quad \therefore \quad \delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}.$$
3. $\psi_{i\alpha}\bar\psi_{l\delta}\bar\psi_{m\rho}\psi_{m\rho}$ なら
$$m=i,\quad m=l,\quad \rho\neq\alpha,\quad \rho\neq\delta;\quad n=j,\quad n=k,\quad \sigma\neq\beta,\quad \sigma\neq\gamma \quad \therefore \quad \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}.$$
4. $\psi_{j\beta}\bar\psi_{k\gamma}\bar\psi_{m\rho}\psi_{m\rho}$ なら
$$m=j,\quad m=k,\quad \rho\neq\beta,\quad \rho\neq\gamma;\quad n=i,\quad n=l,\quad \sigma\neq\alpha,\quad \sigma\neq\delta \quad \therefore \quad \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}.$$

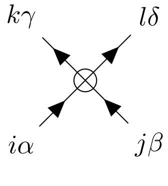
以上から，



$$= ig^2\left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\right).$$

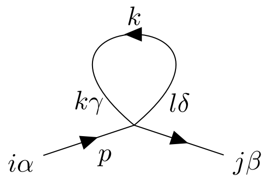
相殺項は

$$i\alpha \xrightarrow{\ p\ } \otimes \text{---} \, j\beta \quad = \left(i\not p\right)_{\beta\alpha}\delta_{ij}\delta_f,$$



$$= 2ig\left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\right)\delta_g.$$

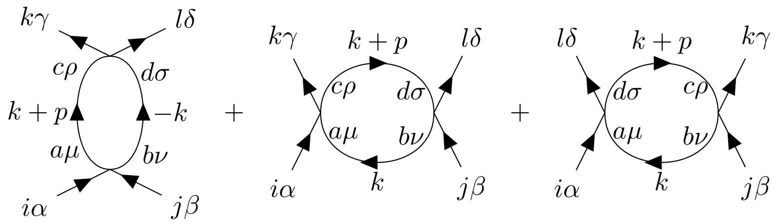
δ_f を求める．



$$= -ig^2 \sum_{kl}\sum_{\gamma\delta}\int \frac{d^dk}{(2\pi)^d}(\cdots)\left(\frac{i}{\not k}\right)_{\gamma\delta}\delta_{kl} = 0$$

なので，$\delta_f = 0$.

δ_g を求める．頂点の 1 ループは



から成る．$\log(-p^2)$ の発散項にのみ興味があるので，それ以外の項は無視する．

1 つ目は

$$V_s = -\frac{(ig^2)^2}{2}\sum_{abcd}\sum_{\mu\nu\rho\sigma}\int \frac{d^dk}{(2\pi)^d}\left(\delta_{kc}\delta_{ld}\delta_{\gamma\rho}\delta_{\delta\sigma} + \delta_{kd}\delta_{lc}\delta_{\gamma\sigma}\delta_{\delta\rho}\right)$$
$$\times\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\delta_{ac}\left(\frac{i}{-\not k}\right)_{\sigma\nu}\delta_{bd}\left(\delta_{ia}\delta_{jb}\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{ib}\delta_{ja}\delta_{\alpha\nu}\delta_{\beta\mu}\right).$$

和を計算すれば

$$\begin{aligned}
&\sum_{abcd}\sum_{\mu\nu\rho\sigma}\left(\delta_{kc}\delta_{ld}\delta_{\gamma\rho}\delta_{\delta\sigma} + \delta_{kd}\delta_{lc}\delta_{\gamma\sigma}\delta_{\delta\rho}\right)\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\delta_{ac}\left(\frac{i}{-\not k}\right)_{\sigma\nu}\delta_{bd} \\
&\quad\times\left(\delta_{ia}\delta_{jb}\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{ib}\delta_{ja}\delta_{\alpha\nu}\delta_{\beta\mu}\right) \\
&= \sum_{ab}\sum_{\mu\nu\rho\sigma}\left(\delta_{ka}\delta_{lb}\delta_{\gamma\rho}\delta_{\delta\sigma} + \delta_{kb}\delta_{la}\delta_{\gamma\sigma}\delta_{\delta\rho}\right)\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\left(\frac{i}{-\not k}\right)_{\sigma\nu} \\
&\quad\times\left(\delta_{ia}\delta_{jb}\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{ib}\delta_{ja}\delta_{\alpha\nu}\delta_{\beta\mu}\right) \\
&= \sum_{ab}\sum_{\mu\nu\rho\sigma}\left(\delta_{ka}\delta_{lb}\delta_{\gamma\rho}\delta_{\delta\sigma}\right)\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\left(\frac{i}{-\not k}\right)_{\sigma\nu}\left(\delta_{ia}\delta_{jb}\delta_{\alpha\mu}\delta_{\beta\nu}\right) \\
&\quad+\sum_{ab}\sum_{\mu\nu\rho\sigma}\left(\delta_{ka}\delta_{lb}\delta_{\gamma\rho}\delta_{\delta\sigma}\right)\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\left(\frac{i}{-\not k}\right)_{\sigma\nu}\left(\delta_{ib}\delta_{ja}\delta_{\alpha\nu}\delta_{\beta\mu}\right) \\
&\quad+\sum_{ab}\sum_{\mu\nu\rho\sigma}\left(\delta_{kb}\delta_{la}\delta_{\gamma\sigma}\delta_{\delta\rho}\right)\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\left(\frac{i}{-\not k}\right)_{\sigma\nu}\left(\delta_{ia}\delta_{jb}\delta_{\alpha\mu}\delta_{\beta\nu}\right) \\
&\quad+\sum_{ab}\sum_{\mu\nu\rho\sigma}\left(\delta_{kb}\delta_{la}\delta_{\gamma\sigma}\delta_{\delta\rho}\right)\left(\frac{i}{\not k + \not p}\right)_{\rho\mu}\left(\frac{i}{-\not k}\right)_{\sigma\nu}\left(\delta_{ib}\delta_{ja}\delta_{\alpha\nu}\delta_{\beta\mu}\right) \\
&= \left(\delta_{ik}\delta_{jl}\right)\left(\frac{1}{\not k + \not p}\right)_{\gamma\alpha}\left(\frac{1}{\not k}\right)_{\delta\beta} + \left(\delta_{il}\delta_{jk}\right)\left(\frac{1}{\not k + \not p}\right)_{\gamma\beta}\left(\frac{1}{\not k}\right)_{\delta\alpha} \\
&\quad+\left(\delta_{il}\delta_{jk}\right)\left(\frac{1}{\not k + \not p}\right)_{\delta\alpha}\left(\frac{1}{\not k}\right)_{\gamma\beta} + \left(\delta_{ik}\delta_{jl}\right)\left(\frac{1}{\not k + \not p}\right)_{\delta\beta}\left(\frac{1}{\not k}\right)_{\gamma\alpha}
\end{aligned}$$

なので,

$$
\begin{aligned}
V_s &= g^4 \left(\delta_{ik}\delta_{jl}\right) \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k}+\not{p})_{\gamma\alpha}(\not{k})_{\delta\beta} + (\not{k})_{\gamma\alpha}(\not{k}+\not{p})_{\delta\beta}}{2k^2(k+p)^2} \\
&\quad + g^4 \left(\delta_{il}\delta_{jk}\right) \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k}+\not{p})_{\gamma\beta}(\not{k})_{\delta\alpha} + (\not{k})_{\gamma\beta}(\not{k}+\not{p})_{\delta\alpha}}{k^2(k+p)^2} \\
&\sim g^4 \delta_{ik}\delta_{jl}(\gamma^\mu)_{\gamma\alpha}(\gamma^\nu)_{\delta\beta} \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^2(k+p)^2} + g^4 \delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma^\nu)_{\delta\alpha} \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^2(k+p)^2} \\
&= g^4 \left[\delta_{ik}\delta_{jl}(\gamma^\mu)_{\gamma\alpha}(\gamma^\nu)_{\delta\beta} + \delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma^\nu)_{\delta\alpha}\right] \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k+p)^2} \frac{k^2 g_{\mu\nu}}{d} \\
&= g^4 \left[\delta_{ik}\delta_{jl}(\gamma^\mu)_{\gamma\alpha}(\gamma_\mu)_{\delta\beta} + \delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma_\mu)_{\delta\alpha}\right] \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k+p)^2} \\
&= g^4 \left[\delta_{ik}\delta_{jl}(\gamma^\mu)_{\gamma\alpha}(\gamma_\mu)_{\delta\beta} + \delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma_\mu)_{\delta\alpha}\right] \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2},
\end{aligned}
$$

$k_\mu k_\nu$ の変換に (A.41) を使った．2 つ目は

$$
\begin{aligned}
V_t = (ig^2)^2 \sum_{abcd} \sum_{\mu\nu\rho\sigma} &\int \frac{d^d k}{(2\pi)^d} \left(\delta_{ik}\delta_{ac}\delta_{\alpha\gamma}\delta_{\mu\rho} + \delta_{ic}\delta_{ak}\delta_{\alpha\rho}\delta_{\mu\gamma}\right) \\
&\times \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \delta_{cd} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \delta_{ab} \left(\delta_{jl}\delta_{db}\delta_{\beta\delta}\delta_{\sigma\nu} + \delta_{jb}\delta_{dl}\delta_{\beta\nu}\delta_{\sigma\delta}\right).
\end{aligned}
$$

和を計算すれば

$$
\begin{aligned}
&\sum_{abcd} \sum_{\mu\nu\rho\sigma} \left(\delta_{ik}\delta_{ac}\delta_{\alpha\gamma}\delta_{\mu\rho} + \delta_{ic}\delta_{ak}\delta_{\alpha\rho}\delta_{\mu\gamma}\right) \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \delta_{cd} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \delta_{ab} \\
&\quad \times \left(\delta_{jl}\delta_{db}\delta_{\beta\delta}\delta_{\sigma\nu} + \delta_{jb}\delta_{dl}\delta_{\beta\nu}\delta_{\sigma\delta}\right) \\
&= \sum_{ac} \sum_{\mu\nu\rho\sigma} \left(\delta_{ik}\delta_{ac}\delta_{\alpha\gamma}\delta_{\mu\rho} + \delta_{ic}\delta_{ak}\delta_{\alpha\rho}\delta_{\mu\gamma}\right) \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \\
&\quad \times \left(\delta_{jl}\delta_{ca}\delta_{\beta\delta}\delta_{\sigma\nu} + \delta_{ja}\delta_{cl}\delta_{\beta\nu}\delta_{\sigma\delta}\right) \\
&= \sum_{ac} \sum_{\mu\nu\rho\sigma} \left(\delta_{ik}\delta_{ac}\delta_{\alpha\gamma}\delta_{\mu\rho}\right) \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \left(\delta_{jl}\delta_{ca}\delta_{\beta\delta}\delta_{\sigma\nu}\right) \\
&\quad + \sum_{ac} \sum_{\mu\nu\rho\sigma} \left(\delta_{ik}\delta_{ac}\delta_{\alpha\gamma}\delta_{\mu\rho}\right) \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \left(\delta_{ja}\delta_{cl}\delta_{\beta\nu}\delta_{\sigma\delta}\right) \\
&\quad + \sum_{ac} \sum_{\mu\nu\rho\sigma} \left(\delta_{ic}\delta_{ak}\delta_{\alpha\rho}\delta_{\mu\gamma}\right) \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \left(\delta_{jl}\delta_{ca}\delta_{\beta\delta}\delta_{\sigma\nu}\right) \\
&\quad + \sum_{ac} \sum_{\mu\nu\rho\sigma} \left(\delta_{ic}\delta_{ak}\delta_{\alpha\rho}\delta_{\mu\gamma}\right) \left(\frac{1}{\not{k}+\not{p}}\right)_{\sigma\rho} \left(\frac{1}{\not{k}}\right)_{\mu\nu} \left(\delta_{ja}\delta_{cl}\delta_{\beta\nu}\delta_{\sigma\delta}\right) \\
&= N \left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}\right) \mathrm{Tr}\left(\frac{1}{\not{k}+\not{p}}\frac{1}{\not{k}}\right) + \left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\right) \left(\frac{1}{\not{k}+\not{p}}\frac{1}{\not{k}}\right)_{\delta\beta} \\
&\quad + \left(\delta_{ik}\delta_{jl}\delta_{\beta\delta}\right) \left(\frac{1}{\not{k}+\not{p}}\frac{1}{\not{k}}\right)_{\gamma\alpha} + \left(\delta_{il}\delta_{jk}\right) \left(\frac{1}{\not{k}}\right)_{\delta\alpha} \left(\frac{1}{\not{k}+\not{p}}\right)_{\gamma\beta} \\
&\sim (2N+2) \left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}\right) \frac{k^2 + k\cdot p}{k^2(k+p)^2} + \left(\delta_{il}\delta_{jk}\right) \left(\frac{1}{\not{k}}\right)_{\delta\alpha} \left(\frac{1}{\not{k}+\not{p}}\right)_{\gamma\beta}
\end{aligned}
$$

なので

$$
\begin{aligned}
V_t &= -g^4(2N+2)\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}\int\frac{d^dk}{(2\pi)^d}\frac{k^2+k\cdot p}{k^2(k+p)^2} - g^4\delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma^\nu)_{\delta\alpha}\int\frac{d^dk}{(2\pi)^d}\frac{k_\mu k_\nu}{k^4} \\
&= -g^4(2N+2)\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}\int\frac{d^dk}{(2\pi)^d}\frac{k^2+k\cdot p}{k^2(k+p)^2} - g^4\delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma_\mu)_{\delta\alpha}\frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}.
\end{aligned}
$$

3 つ目は V_t で $k\gamma \leftrightarrow l\delta$ を入れ替えたもの：

$$
V_u = -g^4(2N+2)\delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\int\frac{d^dk}{(2\pi)^d}\frac{k^2+k\cdot p}{k^2(k+p)^2} - g^4\delta_{ik}\delta_{jl}(\gamma^\mu)_{\delta\beta}(\gamma_\mu)_{\gamma\alpha}\frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}.
$$

以上から

$$
\begin{aligned}
V_s+V_t+V_u &= -g^4(2N+2)\left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\right)\int\frac{d^dk}{(2\pi)^d}\frac{k^2+p\cdot k}{k^2(k+p)^2} \\
&= -g^4(2N+2)\left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\right)\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2+\Delta}{(\ell^2-\Delta)^2} \\
&\approx ig^4(2N+2)\left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\right)\int_0^1 dx\frac{1}{(4\pi)^{d/2}}\frac{d}{2}\Gamma(1-d/2)\left(\frac{1}{\Delta}\right)^{1-d/2}.
\end{aligned}
$$

$\epsilon = 2-d$ とすれば

$$
\begin{aligned}
\frac{1}{(4\pi)^{d/2}}\frac{d}{2}\Gamma(1-d/2)\left(\frac{1}{\Delta}\right)^{1-d/2} &= \frac{1}{4\pi}(1-\epsilon/2)\Gamma(\epsilon/2)\left(\frac{4\pi}{\Delta}\right)^{\epsilon/2} \\
&\approx \frac{1}{4\pi}\left(1-\frac{\epsilon}{2}\right)\left(\frac{2}{\epsilon}-\gamma\right)\left(1-\frac{\epsilon}{2}\log\frac{\Delta}{4\pi}\right) \\
&\approx \frac{1}{4\pi}\left(\frac{2}{\epsilon}-\gamma-\log\frac{\Delta}{4\pi}\right) \\
&\approx -\frac{1}{4\pi}\log(-p^2)
\end{aligned}
$$

なので,

$$
V_s+V_t+V_u = -\frac{ig^4}{4\pi}(2N+2)\left(\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}\right)\log(-p^2).
$$

$p^2=-M^2$ で，これと相殺項の和が 0 なので,

$$
\delta_g = \frac{g^3}{4\pi}(N+1)\log M^2 = \frac{g^3}{2\pi}(N+1)\log M.
$$

Chapter 13

Critical Exponents and Scalar Field Theory

13.3 The Nonlinear Sigma Model

Figure 13.1

非線形シグマ模型 (13.73):

$$\mathcal{L} = \frac{1}{2g^2}\left|\partial_\mu \vec{\pi}\right| + \frac{1}{2g^2}\left(\vec{\pi}\cdot\partial_\mu\vec{\pi}\right)^2$$

の 4 点相関函数を求める．伝播函数は

$$\begin{aligned}
\overbracket{\pi^i(x)\pi^j}(y) &= \int \frac{d^2k}{(2\pi)^2}\frac{ig^2}{p^2}\delta^{ij}e^{-ik\cdot(x-y)},\\
\overbracket{\pi^i(p)\pi^j}(q) &= (2\pi)^2\,\delta^{(2)}(p+q)\frac{ig^2}{p^2}\delta^{ij}.
\end{aligned}$$

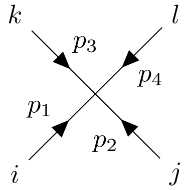
(4.31) から

$$\begin{aligned}
&\langle\Omega|\,T\left\{\pi^i(x_1)\pi^j(x_2)\pi^k(x_3)\pi^l(x_4)\right\}|\Omega\rangle\\
&\approx \frac{i}{2g^2}\sum_{rs}\int d^2x\,\langle 0|\,T\left\{\pi^i(x_1)\pi^j(x_2)\pi^k(x_3)\pi^l(x_4)\pi^r(x)\partial_\mu\pi^r(x)\pi^s(x)\partial^\mu\pi^s(x)\right\}|0\rangle\\
&= \frac{i}{2g^2}\left(\prod_{i=1}^4\frac{d^2p_i}{(2\pi)^2}e^{-ip_i\cdot x_i}\right)\left(\prod_{i=1}^4\frac{d^2k_i}{(2\pi)^2}\right)\sum_{rs}\int d^2x e^{-i(k_1+k_2+k_3+k_4)\cdot x}\\
&\quad\times\langle 0|\,T\left\{\pi^i(p_1)\pi^j(p_2)\pi^k(p_3)\pi^l(p_4)\pi^r(k_1)(-ik_{2\mu})\pi^r(k_2)\pi^s(k_3)(-ik_4^\mu)\pi^s(k_4)\right\}|0\rangle\\
&= -\frac{i}{2g^2}\left(\prod_{i=1}^4\frac{d^2p_i}{(2\pi)^2}e^{-ip_i\cdot x_i}\right)\left(\prod_{i=1}^4\frac{d^2k_i}{(2\pi)^2}\right)(k_2\cdot k_4)\sum_{rs}(2\pi)^2\,\delta^{(2)}(k_1+k_2+k_3+k_4)\\
&\quad\times\langle 0|\,T\left\{\pi^i(p_1)\pi^j(p_2)\pi^k(p_3)\pi^l(p_4)\pi^r(k_1)\pi^r(k_2)\pi^s(k_3)\pi^s(k_4)\right\}|0\rangle\,.
\end{aligned}$$

縮約は全部で 24 通りある．例えば，$\delta^{ij}\delta^{kl}$ を与える項は (p_1,p_2) と (k_1,k_2) を縮約する（4 通り）か，(p_1,p_2) と (k_3,k_4) を縮約する（4 通り），合計 8 通り．伝播函数を代入すれば

$$-\frac{i}{g^2}\left(\prod_{i=1}^4\frac{d^2p_i}{(2\pi)^2}e^{-ip_i\cdot x_i}\frac{ig^2}{{p_i}^2}\right)(2\pi)^2\,\delta^{(2)}(p_1+p_2+p_3+p_4)\delta^{ij}\delta^{kl}(p_1+p_2)(p_3+p_4)$$

を得る．$\delta^{ik}\delta^{jl}$, $\delta^{il}\delta^{jk}$ についても同様に計算すれば



$$= -\frac{i}{g^2}\left[(p_1+p_2)(p_3+p_4)\delta^{ij}\delta^{kl} + (p_1+p_3)(p_2+p_4)\delta^{ik}\delta^{jl} + (p_1+p_4)(p_2+p_3)\delta^{il}\delta^{jk}\right].$$

(13.96)

(7.81) を使えば

$$\begin{aligned}
\langle\phi_a(0)\phi_b(0)\rangle &= \int_{\substack{b\Lambda\le|p|<\Lambda \\ b\Lambda\le|q|<\Lambda}} \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\langle\phi_a(p)\phi_b(q)\rangle = \int_{\substack{b\Lambda\le|p|<\Lambda \\ b\Lambda\le|q|<\Lambda}} \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}(2\pi)^2\,\delta^{(2)}(p+q)\frac{g^2}{p^2}\delta_{ab} \\
&= \int_{b\Lambda\le|p|<\Lambda} \frac{d^2p}{(2\pi)^2}\frac{g^2}{p^2}\delta_{ab} = \int_{b\Lambda\le|p|<\Lambda} \frac{d|p|}{2\pi}\frac{g^2}{|p|}\delta_{ab} = \delta_{ab}\frac{g^2}{2\pi}\log\frac{1}{b}.
\end{aligned}$$

(13.109)

$\beta(T)$ を $T\sim T_*$ で展開して

$$\beta(T) \approx \left[\left.\frac{d\beta}{dT}\right|_{T=T_*}\right](T-T_*).$$

(12.73) から

$$\frac{\partial\bar{T}}{\partial\log p/M} = \beta(\bar{T}) \approx \left[\left.\frac{d\beta}{dT}\right|_{T=T_*}\right](\bar{T}-T_*) = \left[\left.\frac{d\beta}{dT}\right|_{T=T_*}\right]\bar{\rho}_T.$$

(13.114)

(13.113) で部分積分を実行して

$$\begin{aligned}
&\exp\left[-\frac{1}{2{g_0}^2}\int d^dx\left\{(\partial_\mu n)^2 + i\alpha(n^2-1)\right\}\right] \\
&= \exp\left[-\frac{1}{2{g_0}^2}\int d^dx\left\{-\vec{n}\cdot(\partial^2\vec{n}) + i\alpha(n^2-1)\right\}\right] \\
&= \exp\left[-\frac{1}{2{g_0}^2}\int d^dx\left\{\vec{n}\cdot(-\partial^2+i\alpha)\vec{n} - i\alpha\right\}\right] \\
&= \exp\left[-\frac{1}{2{g_0}^2}\int d^dx\,\vec{n}\cdot(-\partial^2+i\alpha)\vec{n}\right]\exp\left[\frac{i}{2{g_0}^2}\int d^dx\,\alpha\right].
\end{aligned}$$

ここで，$\vec{n}$ を適当に変換して

$$Z = \int\mathcal{D}\alpha\mathcal{D}\vec{n}\exp\left[-\int d^dx\,\vec{n}\cdot(-\partial^2+i\alpha)\vec{n}\right]\exp\left[\frac{i}{2{g_0}^2}\int d^dx\,\alpha\right]$$

としてよい．(9.24) から

$$
\begin{aligned}
&\int \mathcal{D}\vec{n}\exp\left[-\int d^dx\,\vec{n}\cdot(-\partial^2+i\alpha)\vec{n}\right]\\
&=\int \mathcal{D}n^1\cdots\mathcal{D}n^N\exp\left[-\int d^dx\,n^1(-\partial^2+i\alpha)n^1\right]\times\cdots\times\exp\left[-\int d^dx\,n^N(-\partial^2+i\alpha)n^N\right]\\
&=[\det(-\partial^2+i\alpha)]^{-N/2}
\end{aligned}
$$

なので，

$$
Z=\int\mathcal{D}\alpha\,[\det(-\partial^2+i\alpha)]^{-N/2}\exp\left[\frac{i}{2{g_0}^2}\int d^dx\,\alpha\right].
$$

2 つ目の表式は (9.77) を使えば得られる．

(13.115)

$(-\partial^2+i\alpha)$ の固有値は固有ベクトル $|k\rangle$ に対し $k^2+i\alpha$ で与えられるので，

$$
\mathrm{Tr}\{\log(-\partial^2+i\alpha)\}=\frac{1}{V}\sum_k\log(k^2+i\alpha).
$$

ここで，(9.22) と同様に離散 Fourier 変換を行った．(13.114) の指数関数の引数が $\alpha(x)$ に関し極小なので，

$$
\begin{aligned}
0&=\frac{\delta}{\delta\alpha(x)}\left[-\frac{N}{2}\mathrm{Tr}\{\log(-\partial^2+i\alpha)\}+\frac{i}{2{g_0}^2}\int d^dy\,\alpha(y)\right]\\
&=\frac{\delta}{\delta\alpha(x)}\left[-\frac{N}{2}\frac{1}{V}\sum_k\log(k^2+i\alpha)\right]+\frac{i}{2{g_0}^2}\int d^dy\,\delta(x-y)\\
&=-\frac{N}{2V}\sum_k\frac{1}{k^2+i\alpha}\frac{\delta}{\delta\alpha(x)}(k^2+i\alpha)+\frac{i}{2{g_0}^2}\\
&=-\frac{iN}{2V}\sum_k\frac{1}{k^2+i\alpha}+\frac{i}{2{g_0}^2}\\
&\to-\frac{iN}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+i\alpha}+\frac{i}{2{g_0}^2}.
\end{aligned}
$$

Problems

Problem 13.3: The $\mathbb{C}P^N$ model

(a)

$\mathbb{C}P^1$ モデルのスカラーを

$$
\boldsymbol{z}=\begin{pmatrix}z_1\\z_2\end{pmatrix}
$$

とする．$\vec{n}=\boldsymbol{z}^\dagger\vec{\sigma}\boldsymbol{z}$ を計算すれば

$$
\begin{aligned}
n_1&=(z_1^*,z_2^*)\sigma^1\begin{pmatrix}z_1\\z_2\end{pmatrix}=(z_1^*,z_2^*)\begin{pmatrix}&1\\1&\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=z_1z_2^*+z_1^*z_2,\\
n_2&=(z_1^*,z_2^*)\sigma^2\begin{pmatrix}z_1\\z_2\end{pmatrix}=(z_1^*,z_2^*)\begin{pmatrix}&-i\\i&\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=i(z_1z_2^*-z_1^*z_2),
\end{aligned}
$$

$$n_3 = (z_1^*, z_2^*)\sigma^3 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1^*, z_2^*) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |z_1|^2 - |z_2|^2.$$

この結果を使えば

$$n_1{}^2 + n_2{}^2 + n_3{}^2 = (|z_1|^2 + |z_2|^2)^2$$

なので，規格化条件 $n_1{}^2 + n_2{}^2 + n_3{}^2 = 1$ は $|z_1|^2 + |z_2|^2 = 1$ と等しい．非線形シグマ模型のラグランジアン (13.67) に代入して

$$\begin{aligned}
&(\partial_\mu n^1)(\partial^\mu n^1) + (\partial_\mu n^2)(\partial^\mu n^2) + (\partial_\mu n^3)(\partial^\mu n^3) \\
&= [\partial_\mu(z_1 z_2^*) + \partial_\mu(z_1^* z_2)][\partial^\mu(z_1 z_2^*) + \partial^\mu(z_1^* z_2)] \\
&\quad - [\partial_\mu(z_1 z_2^*) - \partial_\mu(z_1^* z_2)][\partial^\mu(z_1 z_2^*) - \partial^\mu(z_1^* z_2)] \\
&\quad + [\partial_\mu(z_1 z_1^*) - \partial_\mu(z_2 z_2^*)][\partial^\mu(z_1 z_1^*) - \partial^\mu(z_2 z_2^*)] \\
&= 4\partial_\mu(z_1 z_2^*)\partial^\mu(z_1^* z_2) + \partial_\mu(z_1 z_1^*)\partial^\mu(z_1 z_1^*) + \partial_\mu(z_2 z_2^*)\partial^\mu(z_2 z_2^*) - 2\partial_\mu(z_1 z_1^*)\partial^\mu(z_2 z_2^*) \\
&= 4|z_1|^2(\partial_\mu z_2)(\partial^\mu z_2^*) + 4|z_2|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + 4z_1 z_2(\partial_\mu z_1^*)(\partial^\mu z_2^*) + 4z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2) \\
&\quad + 2|z_1|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + z_1{}^2(\partial_\mu z_1^*)(\partial^\mu z_1^*) + z_1^{*2}(\partial_\mu z_1)(\partial^\mu z_1) \\
&\quad + 2|z_2|^2(\partial_\mu z_2)(\partial^\mu z_2^*) + z_2{}^2(\partial_\mu z_2^*)(\partial^\mu z_2^*) + z_2^{*2}(\partial_\mu z_2)(\partial^\mu z_2) \\
&\quad - 2z_1 z_2(\partial_\mu z_1^*)(\partial^\mu z_2^*) - 2z_1^* z_2(\partial_\mu z_1)(\partial^\mu z_2^*) - 2z_1 z_2^*(\partial_\mu z_1^*)(\partial^\mu z_2) - 2z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2) \\
&= 4|z_1|^2(\partial_\mu z_2)(\partial^\mu z_2^*) + 4|z_2|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + 2|z_1|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + 2|z_2|^2(\partial_\mu z_2)(\partial^\mu z_2^*) \\
&\quad + z_1{}^2(\partial_\mu z_1^*)(\partial^\mu z_1^*) + z_1^{*2}(\partial_\mu z_1)(\partial^\mu z_1) + z_2{}^2(\partial_\mu z_2^*)(\partial^\mu z_2^*) + z_2^{*2}(\partial_\mu z_2)(\partial^\mu z_2) \\
&\quad + 2z_1 z_2(\partial_\mu z_1^*)(\partial^\mu z_2^*) + 2z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2) - 2z_1^* z_2(\partial_\mu z_1)(\partial^\mu z_2^*) - 2z_1 z_2^*(\partial_\mu z_1^*)(\partial^\mu z_2) \\
&= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 2|z_1|^2(\partial_\mu z_1)(\partial^\mu z_1^*) - 2|z_2|^2(\partial_\mu z_2)(\partial^\mu z_2^*) \\
&\quad - 2z_1^* z_2(\partial_\mu z_1)(\partial^\mu z_2^*) - 2z_1 z_2^*(\partial_\mu z_1^*)(\partial^\mu z_2) \\
&\quad + (z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1^*\partial^\mu z_1 + z_2^*\partial^\mu z_2) + (z_1\partial_\mu z_1^* + z_2\partial_\mu z_2^*)(z_1\partial^\mu z_1^* + z_2\partial^\mu z_2^*) \\
&= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 2(z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1\partial^\mu z_1^* + z_2\partial^\mu z_2^*) \\
&\quad + (z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1^*\partial^\mu z_1 + z_2^*\partial^\mu z_2) + (z_1\partial_\mu z_1^* + z_2\partial_\mu z_2^*)(z_1\partial^\mu z_1^* + z_2\partial^\mu z_2^*).
\end{aligned}$$

$|z_1|^2 + |z_2|^2 = 1$ を微分して

$$z_1\partial_\mu z_1^* + z_1^*\partial_\mu z_1 + z_2\partial_\mu z_2^* + z_2^*\partial_\mu z_2 = 0$$

なので,

$$\begin{aligned}
&= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 2(z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1\partial^\mu z_1^* + z_2\partial^\mu z_2^*) \\
&\quad + 2(z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1^*\partial^\mu z_1 + z_2^*\partial^\mu z_2) \\
&= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) \\
&\quad - 2(z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1\partial^\mu z_1^* + z_2\partial^\mu z_2^* - z_1^*\partial^\mu z_1 - z_2^*\partial^\mu z_2) \\
&= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 4(z_1^*\partial_\mu z_1 + z_2^*\partial_\mu z_2)(z_1\partial^\mu z_1^* + z_2\partial^\mu z_2^*).
\end{aligned}$$

以上から

$$|\partial_\mu \vec{n}|^2 = 4\left[(\partial_\mu \boldsymbol{z})(\partial^\mu \boldsymbol{z}^*) + (\boldsymbol{z}^* \cdot \partial_\mu \boldsymbol{z})(\boldsymbol{z} \cdot \partial^\mu \boldsymbol{z}^*)\right].$$

Final Project II: The Coleman-Weinberg Potential

(a)

Coleman-Weinberg 模型：

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger(D_\mu\phi) + \mu^2\phi^\dagger\phi - \frac{\lambda}{6}(\phi^\dagger\phi)^2.$$

ポテンシャル項は

$$\phi_0 = \mu\sqrt{\frac{3}{\lambda}}$$

で最小値を取る．

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}[\sigma(x) + i\pi(x)]$$

とする．

$$\begin{aligned}
D_\mu\phi(x) &= (\partial_\mu + ieA_\mu)\left[\phi_0 + \frac{1}{\sqrt{2}}[\sigma(x) + i\pi(x)]\right] \\
&= \frac{1}{\sqrt{2}}\partial_\mu\sigma + \frac{i}{\sqrt{2}}\partial_\mu\pi + ieA_\mu\phi_0 + \frac{i}{\sqrt{2}}eA_\mu\sigma(x) - \frac{1}{\sqrt{2}}eA_\mu\pi(x) \\
&= \left[\frac{1}{\sqrt{2}}\partial_\mu\sigma - \frac{1}{\sqrt{2}}eA_\mu\pi(x)\right] + \frac{i}{\sqrt{2}}\partial_\mu\pi + ieA_\mu\phi_0 + \frac{i}{\sqrt{2}}eA_\mu\sigma(x)
\end{aligned}$$

なので，

$$\begin{aligned}
(D_\mu\phi)^\dagger(D_\mu\phi) &= \left[\frac{1}{\sqrt{2}}\partial_\mu\sigma - \frac{1}{\sqrt{2}}eA_\mu\pi(x)\right]^2 + \left[\frac{1}{\sqrt{2}}\partial_\mu\pi + eA_\mu\phi_0 + \frac{1}{\sqrt{2}}eA_\mu\sigma(x)\right]^2 \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}e^2A_\mu A^\mu\pi^2 - e\pi A^\mu\partial_\mu\sigma \\
&\quad + \frac{1}{2}(\partial_\mu\pi)^2 + {\phi_0}^2e^2A_\mu A^\mu + \frac{1}{2}e^2A_\mu A^\mu\sigma^2 \\
&\quad + \sqrt{2}\phi_0 eA^\mu\partial_\mu\pi + \sqrt{2}\phi_0 e^2A^\mu A_\mu\sigma + e\sigma A^\mu\partial_\mu\pi \\
&= \frac{1}{2}(\sqrt{2}\phi_0)^2e^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}e^2A_\mu A^\mu\pi^2 + \frac{1}{2}e^2A_\mu A^\mu\sigma^2 \\
&\quad + e\sigma A^\mu\partial_\mu\pi - e\pi A^\mu\partial_\mu\sigma + \sqrt{2}\phi_0 eA^\mu\partial_\mu\pi + \sqrt{2}\phi_0 e^2A^\mu A_\mu\sigma.
\end{aligned}$$

さらに

$$\phi^\dagger\phi = \left(\phi_0 + \frac{\sigma}{\sqrt{2}}\right)^2 + \left(\frac{\pi}{\sqrt{2}}\right)^2 = {\phi_0}^2 + \sqrt{2}\phi_0\sigma + \frac{\sigma^2}{2} + \frac{\pi^2}{2}$$

および

$$
\begin{aligned}
(\phi^\dagger\phi)^2 &= \left[\left(\phi_0 + \frac{\sigma}{\sqrt{2}}\right)^2 + \left(\frac{\pi}{\sqrt{2}}\right)^2\right]^2 \\
&= \left(\phi_0 + \frac{\sigma}{\sqrt{2}}\right)^4 + \pi^2\left(\phi_0 + \frac{\sigma}{\sqrt{2}}\right)^2 + \frac{\pi^4}{4} \\
&= {\phi_0}^4 + (\sqrt{2}\phi_0)^3\sigma + 3{\phi_0}^2\sigma^2 + \sqrt{2}\phi_0\sigma^3 + \frac{\sigma^4}{4} + \pi^2\left({\phi_0}^2 + \sqrt{2}\phi_0\sigma + \frac{\sigma^2}{2}\right) + \frac{\pi^4}{4}.
\end{aligned}
$$

以上から

$$
\begin{aligned}
\mathcal{L} \to & -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\sqrt{2}\phi_0)^2 e^2 A_\mu A^\mu \\
& + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}e^2 A_\mu A^\mu \pi^2 + \frac{1}{2}e^2 A_\mu A^\mu \sigma^2 \\
& + e\sigma A^\mu\partial_\mu\pi - e\pi A^\mu\partial_\mu\sigma + \sqrt{2}\phi_0 e A^\mu\partial_\mu\pi + \sqrt{2}\phi_0 e^2 A^\mu A_\mu\sigma \\
& + \mu^2\left(\sqrt{2}\phi_0\sigma + \frac{\sigma^2}{2} + \frac{\pi^2}{2}\right) \\
& - \frac{\lambda}{6}\left[(\sqrt{2}\phi_0)^3\sigma + 3{\phi_0}^2\sigma^2 + \sqrt{2}\phi_0\sigma^3 + \frac{\sigma^4}{4} + \pi^2\left({\phi_0}^2 + \sqrt{2}\phi_0\sigma + \frac{\sigma^2}{2}\right) + \frac{\pi^4}{4}\right] \\
= & -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}\left(\frac{6\mu^2 e^2}{\lambda}\right) A_\mu A^\mu + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}(\partial_\mu\pi)^2 \\
& - \frac{\lambda}{6}\left[\frac{\sigma^4}{4} + \mu\sqrt{\frac{6}{\lambda}}\sigma^3 + \frac{\pi^2\sigma^2}{2} + \mu\sqrt{\frac{6}{\lambda}}\pi^2\sigma + \frac{\pi^4}{4}\right] \\
& + \frac{1}{2}e^2 A_\mu A^\mu \pi^2 + \frac{1}{2}e^2 A_\mu A^\mu \sigma^2 \\
& + e\sigma A^\mu\partial_\mu\pi - e\pi A^\mu\partial_\mu\sigma + \mu\sqrt{\frac{6}{\lambda}} e A^\mu\partial_\mu\pi + \mu\sqrt{\frac{6}{\lambda}} e^2 A^\mu A_\mu\sigma.
\end{aligned}
$$

Part III

Non-Abelian Gauge Theories

Chapter 15

Non-Abelian Gauge Invariance

15.3 The Gauge-Invariant Wilson Loop

(15.62)

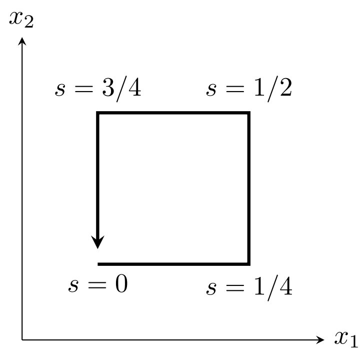
(15.56) を ϵ の 1 次まで展開すれば，

$$U_P(x,x) \approx 1 + ig \oint ds \, \frac{dx^\mu}{ds} A^a_\mu(x(s)) t^a \approx 1 + ig \left[\frac{\partial A^a_2}{\partial x^1} \epsilon^2 - \frac{\partial A^a_1}{\partial x^2} \epsilon^2 \right] t^a.$$

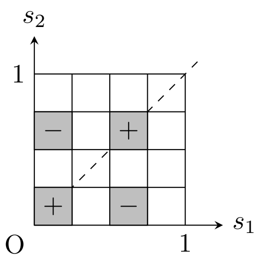
2 次の展開は

$$\begin{aligned} & -\frac{g^2}{2} \int_0^1 ds_1 \int_0^1 ds_2 \, \frac{dx^\mu}{ds_1} \frac{dx^\nu}{ds_2} A^a_\mu(x(s_1)) A^b_\nu(x(s_1)) P\{t^a t^b\} \\ & \approx -\frac{g^2}{2} A^a_\mu(x) A^b_\nu(x) \int_0^1 ds_1 \int_0^1 ds_2 \, \frac{dx^\mu}{ds_1} \frac{dx^\nu}{ds_2} P\{t^a t^b\} \end{aligned}$$

で与えられる．$s_1 > s_2$ なら $P\{t^a t^b\} = t^a t^b$ である．



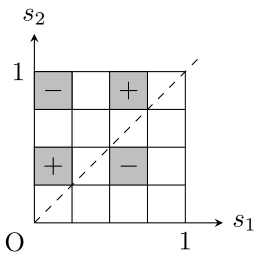
$\mu = \nu = 1$ の場合，$dx^1/ds \neq 0$ となるのは $0 < s < 1/4$ と $1/2 < s < 3/4$ のみ．



$s_1 > s_2$ の領域では積分が 0 になり，$s_1 < s_2$ の領域でも同様に積分は 0 となる．

$\mu = \nu = 2$ の場合，$dx^2/ds \neq 0$ となるのは $1/4 < s < 1/2$ と $3/4 < s < 1$ のみ．$s_1 > s_2$ の領域では積分が 0 になり，$s_1 < s_2$ の領域でも同様に積分は 0 となる．

$\mu = 1, \nu = 2$ の場合．



$s_1 > s_2$ の領域では積分は

$$-16\epsilon^2 \frac{1}{16} t^a t^b = -\epsilon^2 t^a t^b.$$

$s_1 > s_2$ の領域では積分は

$$+16\epsilon^2 \frac{1}{16} t^b t^a = \epsilon^2 t^b t^a.$$

(15.44) から

$$\epsilon^2 \frac{g^2}{2} A_1^a A_2^b [t^a, t^b] = \frac{i}{2} g^2 \epsilon^2 f^{abc} A_1^a A_2^b t^c.$$

$\mu = 2, \nu = 1$ の場合も同様なので，

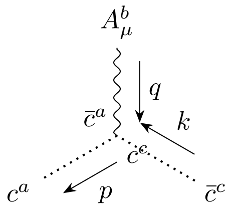
$$\begin{aligned} U_P(x, x) &\approx 1 + ig \left[\frac{\partial A_2^a}{\partial x^1} \epsilon^2 - \frac{\partial A_1^a}{\partial x^2} \epsilon^2 \right] t^a + ig\epsilon^2 f^{abc} A_1^a A_2^b t^c \\ &= 1 + ig\epsilon^2 \left[\partial_1 A_2^c - \partial_2 A_1^c + g f^{abc} A_1^a A_2^b \right] t^c. \end{aligned}$$

Chapter 16

Quantization of Non-Abelian Gauge Theories

16.3 Ghosts and Unitarity

Figure 16.5



(16.32) から

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= \bar{c}^a\left(-g\partial^\mu f^{abc}A^b_\mu\right)c^c + \cdots \\ &= -gf^{abc}\bar{c}^a(\partial^\mu A^b_\mu)c^c - gf^{abc}\bar{c}^a A^b_\mu(\partial^\mu c^c) + \cdots \\ &= -gf^{abc}\bar{c}^a(-iq^\mu A^b_\mu)c^c - gf^{abc}\bar{c}^a A^b_\mu(-ik^\mu c^c) + \cdots \\ &= -gf^{abc}\bar{c}^a(-ip^\mu A^b_\mu)c^c + \cdots\end{aligned}$$

となるので，頂点は

$$i\mathcal{L} \to -gf^{abc}p_\mu.$$

16.5 One-Loop Divergences of Non-Abelian Gauge Theory

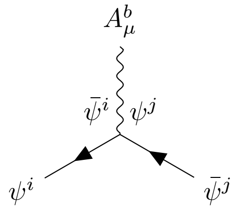
p.522 最後の式

フェルミオンは $d(r)$ 個の Dirac スピノル ψ_i からなる：$\psi = (\psi_1, \ldots, \psi_{d(r)})$. Lie 代数の生成子はサイズが $d(r) \times d(r)$ の行列 $d(G)$ 個の集合：$\{\, t^a \mid 1 \leq a \leq d(G) \,\}$.

(16.34) のうち，ボソンとフェルミオンの頂点は

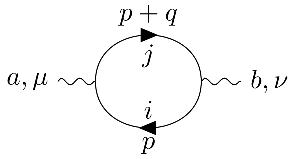
$$\mathcal{L}_{\bar{\psi}\psi A} = g\bar{\psi}^i\gamma^\mu\psi^j A^b_\mu (t^b)_{ij}$$

なので，



$$= ig\gamma^\mu (t^b)_{ij}$$

で与えられる．よって，



$$\begin{aligned}
&= -(ig)^2 \int \frac{d^dp}{(2\pi)^d} \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not{q}} \gamma^\nu \frac{i}{\not{p}+\not{q}}\right] (t^a)_{ji} (t^b)_{ij} \\
&= -(ig)^2 \int \frac{d^dp}{(2\pi)^d} \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not{q}} \gamma^\nu \frac{i}{\not{p}+\not{q}}\right] \mathrm{Tr}[t^a t^b].
\end{aligned}$$

(16.72)

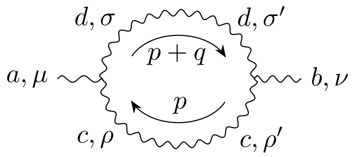
一般ゲージ (16.29)

$$\overbrace{A^a_\mu(p) A^b_\nu}(q) = (2\pi)^4 \delta^{(4)}(p+q) \frac{-i}{p^2} \left[g_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right] \delta^{ab}.$$

を使って計算する．

(16.62) の修正

(16.60) は



$$\begin{aligned}
&= \frac{1}{2} \int \frac{d^dp}{(2\pi)^d} \frac{-i}{p^2} \frac{-i}{(p+q)^2} g^2 f^{acd} f^{bcd} N^{\mu\nu}_\xi \\
&= -\frac{g^2}{2} C_2(G) \delta^{ab} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2 (p+q)^2} N^{\mu\nu}_\xi
\end{aligned}$$

となる．ただし，

$$\begin{aligned}
N^{\mu\nu}_\xi = &\left[g_{\sigma\sigma'} - (1-\xi) \frac{(p+q)_\sigma (p+q)_{\sigma'}}{(p+q)^2} \right] \left[g_{\rho\rho'} - (1-\xi) \frac{p_\rho p_{\rho'}}{p^2} \right] \\
&\times \left[g^{\mu\rho} (q-p)^\sigma + g^{\rho\sigma} (2p+q)^\mu + g^{\sigma\mu} (-p-2q)^\rho \right] \\
&\quad \times \left[g^{\nu\rho'} (p-q)^{\sigma'} + g^{\rho'\sigma'} (-2p-q)^\nu + g^{\sigma'\nu} (p+2q)^{\rho'} \right] \\
= &N^{\mu\nu} \\
&- g_{\sigma\sigma'} (1-\xi) \frac{p_\rho p_{\rho'}}{p^2} [\cdots][\cdots] \\
&- g_{\rho\rho'} (1-\xi) \frac{(p+q)_\sigma (p+q)_{\sigma'}}{(p+q)^2} [\cdots][\cdots] \\
&+ (1-\xi)^2 \frac{(p+q)_\sigma (p+q)_{\sigma'}}{(p+q)^2} \frac{p_\rho p_{\rho'}}{p^2} [\cdots][\cdots]
\end{aligned}$$

$$
\begin{aligned}
&= N^{\mu\nu} \\
&\quad - \frac{1-\xi}{p^2} \left[p^\mu (q-p)^\sigma + p^\sigma (2p+q)^\mu + g^{\sigma\mu} p\cdot(-p-2q) \right] \\
&\qquad \times \left[p^\nu (p-q)_\sigma + p_\sigma (-2p-q)^\nu + \delta^\nu{}_\sigma p\cdot(p+2q) \right] \\
&\quad - \frac{1-\xi}{(p+q)^2} \left[g^{\mu\rho} (p+q)\cdot(q-p) + (p+q)^\rho (2p+q)^\mu + (p+q)^\mu (-p-2q)^\rho \right] \\
&\qquad \times \left[\delta^\nu{}_\rho (p+q)\cdot(p-q) + (p+q)_\rho (-2p-q)^\nu + (p+q)^\nu (p+2q)_\rho \right] \\
&\quad + \frac{(1-\xi)^2}{p^2(p+q)^2} \left[p^\mu (p+q)\cdot(q-p) + p\cdot(p+q)(2p+q)^\mu + (p+q)^\mu p\cdot(-p-2q) \right] \\
&\qquad \times \left[p^\nu (p+q)\cdot(p-q) + p\cdot(p+q)(-2p-q)^\nu + (p+q)^\nu p\cdot(p+2q) \right] \\
&= N^{\mu\nu} \\
&\quad - \frac{1-\xi}{p^2} \left[p^\mu (q-p)^\sigma + p^\sigma (2p+q)^\mu + g^{\sigma\mu} p\cdot(-p-2q) \right] \\
&\qquad \times \left[p^\nu (p-q)_\sigma + p_\sigma (-2p-q)^\nu + \delta^\nu{}_\sigma p\cdot(p+2q) \right] \\
&\quad - \frac{1-\xi}{(p+q)^2} \left[(p+q)^\mu (-p-2q)^\sigma + (p+q)^\sigma (2p+q)^\mu + g^{\sigma\mu} (p+q)\cdot(q-p) \right] \\
&\qquad \times \left[(p+q)^\nu (p+2q)_\sigma + (p+q)_\sigma (-2p-q)^\nu + \delta^\nu{}_\sigma (p+q)\cdot(p-q) \right] \\
&\quad + \frac{(1-\xi)^2}{p^2(p+q)^2} \left[p^\mu (p+q)\cdot(q-p) + p\cdot(p+q)(2p+q)^\mu + (p+q)^\mu p\cdot(-p-2q) \right] \\
&\qquad \times \left[p^\nu (p+q)\cdot(p-q) + p\cdot(p+q)(-2p-q)^\nu + (p+q)^\nu p\cdot(p+2q) \right].
\end{aligned}
$$

ここで，第 2 項の $[\cdots][\cdots]$ を $p \mapsto p+q$ および $q \mapsto -q$ とすれば第 3 項の $[\cdots][\cdots]$ となる．さらに，

$$
\int \frac{d^dp}{(2\pi)^d} \frac{1}{p^4(p+q)^2} \mapsto \int \frac{d^dp}{(2\pi)^d} \frac{1}{(p+q)^4 p^2}
$$

となるので，第 2 項と第 3 項の積分は等しい．

第 4 項を計算する．(6.42) より分母は

$$
\begin{aligned}
\int \frac{d^dp}{(2\pi)^d} \frac{1}{p^4(p+q)^4} &= 6 \int_0^1 dx\, dy\, \delta(x+y-1) \int \frac{d^dp}{(2\pi)^d} \frac{xy}{[yp^2 + x(p+q)^2]^4} \\
&= 6 \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{x(1-x)}{[(1-x)p^2 + x(p+q)^2]^4} \\
&= 6 \int_0^1 dx \int \frac{d^dP}{(2\pi)^d} \frac{x(1-x)}{[P^2 - \Delta]^4} \\
&\quad (P = p + xq, \quad \Delta = -x(1-x)q^2).
\end{aligned}
$$

分子は

$$
\begin{aligned}
&p^\mu (p+q)\cdot(q-p) + p\cdot(p+q)(2p+q)^\mu + (p+q)^\mu p\cdot(-p-2q) \\
&= p^\mu (q^2 - p^2) + (p^2 + p\cdot q)(2p+q)^\mu + (p+q)^\mu (-p^2 - 2p\cdot q) \\
&= q^2 p^\mu - (p\cdot q) q^\mu
\end{aligned}
$$

及び

$$
\begin{aligned}
&p^\nu (p+q)\cdot(p-q) + p\cdot(p+q)(-2p-q)^\nu + (p+q)^\nu p\cdot(p+2q) \\
&= p^\nu (p^2 - q^2) + (p^2 + p\cdot q)(-2p-q)^\nu + (p+q)^\nu (p^2 + 2p\cdot q)
\end{aligned}
$$

$$= -q^2 p^\nu + (p\cdot q) q^\nu$$

をかけて

$$\begin{aligned}
&[q^2 p^\mu - (p\cdot q)q^\mu][-q^2 p^\nu + (p\cdot q)q^\nu] \\
&= -q^4 p^\mu p^\nu - (p\cdot q)^2 q^\mu q^\nu + q^2 (p\cdot q)(p^\mu q^\nu + q^\mu p^\nu) \\
&\to -q^4\left[P^\mu P^\nu + x^2 q^\mu q^\nu\right] - \left[P\cdot q - xq^2\right]^2 q^\mu q^\nu \\
&\quad + q^2[P\cdot q - xq^2][P^\mu q^\nu + q^\mu P^\nu - 2xq^\mu q^\nu] \\
&\to -q^4\left[P^\mu P^\nu + x^2 q^\mu q^\nu\right] - \left[(P\cdot q)^2 + x^2 q^4\right] q^\mu q^\nu \\
&\quad + q^2\left[P^\rho q_\rho (P^\mu q^\nu + q^\mu P^\nu) + 2x^2 q^2 q^\mu q^\nu\right] \\
&= -q^4 P^\mu P^\nu - x^2 q^4 q^\mu q^\nu - \left[(P\cdot q)^2 + x^2 q^4\right] q^\mu q^\nu \\
&\quad + q^2 q^\nu q_\rho P^\mu P^\rho + q^2 q^\mu q_\rho P^\nu P^\rho + 2x^2 q^4 q^\mu q^\nu \\
&= -q^4 P^\mu P^\nu - q_\rho q_\sigma q^\mu q^\nu P^\rho P^\sigma + q^2 q^\nu q_\rho P^\mu P^\rho + q^2 q^\mu q_\rho P^\nu P^\rho \\
&\to -q^4 \frac{g^{\mu\nu}}{d} P^2 - q_\rho q_\sigma q^\mu q^\nu \frac{g^{\rho\sigma}}{d} P^2 + q^2 q^\nu q_\rho \frac{g^{\mu\rho}}{d} P^2 + q^2 q^\mu q_\rho \frac{g^{\nu\rho}}{d} P^2 \\
&= -q^4 \frac{g^{\mu\nu}}{d} P^2 - q^2 q^\mu q^\nu \frac{1}{d} P^2 + q^2 q^\nu q^\mu \frac{1}{d} P^2 + q^2 q^\mu q^\nu \frac{1}{d} P^2 \\
&= -(q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{q^2}{d} P^2 .
\end{aligned}$$

よって，第 4 項は

$$-3g^2(1-\xi)^2 C_2(G)\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu)\frac{q^2}{d}\int_0^1 dx\, x(1-x)\int \frac{d^d P}{(2\pi)^d}\frac{P^2}{[P^2-\Delta]^4} \qquad [16.5.1]$$

となる（有限値）．

第 2 項（と第 3 項は等しい）を計算する．(6.40) から分母は

$$\begin{aligned}
&\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^4 (p+q)^2} \\
&= \int_0^1 dx\, dy\, \delta(x+y-1)\int \frac{d^d p}{(2\pi)^d}\frac{2y}{[(1-x)p^2 + x(p+q)^2]^3} \\
&= 2\int_0^1 dx\,(1-x)\int \frac{d^d P}{(2\pi)^d}\frac{1}{[P^2-\Delta]^3}, \\
&(P = p + xq, \quad \Delta = -x(1-x)q^2).
\end{aligned}$$

分子は

$$\begin{aligned}
&[p^\mu (q-p)^\sigma + p^\sigma (2p+q)^\mu + g^{\sigma\mu} p\cdot(-p-2q)] \\
&\times [p^\nu (p-q)_\sigma + p_\sigma(-2p-q)^\nu + \delta^\nu{}_\sigma p\cdot(p+2q)] \\
&= -p^\mu p^\nu (q-p)^2 - g^{\mu\nu}(p^2 + 2p\cdot q)^2 - p^2 (q+2p)^\mu (q+2p)^\nu \\
&\qquad + p\cdot(p-q)(2p+q)^\mu p^\nu + p\cdot(p+2q)(q-p)^\mu p^\nu + p\cdot(p+2q)p^\mu(2p+q)^\nu \\
&\qquad + ((\mu \leftrightarrow \nu)) \\
&= -(P^\mu - xq^\mu)(P^\nu - xq^\nu)[P-(x+1)q]^2 \\
&\quad - g^{\mu\nu}\left[(P-xq)^2 + 2(P-xq)\cdot q\right]^2 \\
&\quad - (P-xq)^2 (2P + (1-2x)q)^\mu (2P + (1-2x)q)^\nu
\end{aligned}$$

$$
\begin{aligned}
&\quad + (P - xq)\cdot(P-(x+1)q)(2P+(1-2x)q)^\mu(P-xq)^\nu \\
&\quad + (P - xq)\cdot(P+(2-x)q)(-P+(x+1)q)^\mu(P-xq)^\nu \\
&\quad + (P - xq)\cdot(P+(2-x)q)(P-xq)^\mu(2P+(1-2x)q)^\nu \\
&\quad + ((\mu \leftrightarrow \nu)) \\
&\to -(P^\mu P^\nu - xP^\mu q^\nu - xq^\mu P^\nu + x^2 q^\mu q^\nu)\left[P^2 - 2(x+1)P\cdot q + (x+1)^2q^2\right] \\
&\ \ - g^{\mu\nu}\left[P^2 - 2(x-1)P\cdot q + x(x-2)q^2\right]^2 \\
&\ \ - (P^2 - 2xP\cdot q + x^2q^2)[4P^\mu P^\nu + 2(1-2x)(P^\mu q^\nu + q^\mu P^\nu) + (1-2x)^2 q^\mu q^\nu] \\
&\quad + (P^2 - (2x+1)P\cdot q + x(x+1)q^2)(2P^\mu P^\nu - 2xP^\mu q^\nu + (1-2x)q^\mu P^\nu - x(1-2x)q^\mu q^\nu) \\
&\quad + (P^2 + 2(1-x)P\cdot q - x(2-x)q^2)(-P^\mu P^\nu + xP^\mu q^\nu + (x+1)q^\mu P^\nu - x(x+1)q^\mu q^\nu) \\
&\quad + (P^2 + 2(1-x)P\cdot q - x(2-x)q^2)(2P^\mu P^\nu + (1-2x)P^\mu q^\nu - 2xq^\mu P^\nu - x(1-2x)q^\mu q^\nu) \\
&\quad + ((\mu \leftrightarrow \nu)) \\
&\to -P^2P^\mu P^\nu - (x+1)^2q^2P^\mu P^\nu - x^2q^\mu q^\nu P^2 \\
&\quad - x^2(x+1)^2q^2q^\mu q^\nu - 2x(x+1)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ - g^{\mu\nu}\left[(P^2)^2 + 4(x-1)^2q_\rho q_\sigma P^\rho P^\sigma + 2x(x-2)q^2P^2 + x^2(x-2)^2q^4\right] \\
&\ \ - 4P^2P^\mu P^\nu - 4x^2q^2P^\mu P^\nu - (1-2x)^2q^\mu q^\nu P^2 \\
&\quad - x^2(1-2x)^2q^2q^\mu q^\nu + 4x(1-2x)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ + (P^2 - (2x+1)P\cdot q + x(x+1)q^2)[4P^\mu P^\nu + (1-4x)(P^\mu q^\nu + q^\mu P^\nu) - 2x(1-2x)q^\mu q^\nu] \\
&\ \ + (P^2 + 2(1-x)P\cdot q - x(2-x)q^2)[-2P^\mu P^\nu + (2x+1)(P^\mu q^\nu + q^\mu P^\nu) - 2x(x+1)q^\mu q^\nu] \\
&\ \ + (P^2 + 2(1-x)P\cdot q - x(2-x)q^2)[4P^\mu P^\nu + (1-4x)(P^\mu q^\nu + q^\mu P^\nu) - 2x(1-2x)q^\mu q^\nu] \\
&= -P^2P^\mu P^\nu - (x+1)^2q^2P^\mu P^\nu - x^2q^\mu q^\nu P^2 \\
&\quad - x^2(x+1)^2q^2q^\mu q^\nu - 2x(x+1)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ - g^{\mu\nu}\left[(P^2)^2 + 4(x-1)^2q_\rho q_\sigma P^\rho P^\sigma + 2x(x-2)q^2P^2 + x^2(x-2)^2q^4\right] \\
&\ \ - 4P^2P^\mu P^\nu - 4x^2q^2P^\mu P^\nu - (1-2x)^2q^\mu q^\nu P^2 \\
&\quad - x^2(1-2x)^2q^2q^\mu q^\nu + 4x(1-2x)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ + (2P^2 + (1-4x)P\cdot q + x(2x-1)q^2)[4P^\mu P^\nu + (1-4x)(P^\mu q^\nu + q^\mu P^\nu) - 2x(1-2x)q^\mu q^\nu] \\
&\ \ + (P^2 + 2(1-x)P\cdot q - x(2-x)q^2)[-2P^\mu P^\nu + (2x+1)(P^\mu q^\nu + q^\mu P^\nu) - 2x(x+1)q^\mu q^\nu] \\
&\to -P^2P^\mu P^\nu - (x+1)^2q^2P^\mu P^\nu - x^2q^\mu q^\nu P^2 \\
&\quad - x^2(x+1)^2q^2q^\mu q^\nu - 2x(x+1)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ - g^{\mu\nu}\left[(P^2)^2 + 4(x-1)^2q_\rho q_\sigma P^\rho P^\sigma + 2x(x-2)q^2P^2 + x^2(x-2)^2q^4\right] \\
&\ \ - 4P^2P^\mu P^\nu - 4x^2q^2P^\mu P^\nu - (1-2x)^2q^\mu q^\nu P^2 \\
&\quad - x^2(1-2x)^2q^2q^\mu q^\nu + 4x(1-2x)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ + 8P^2P^\mu P^\nu + 4x(2x-1)q^2P^\mu P^\nu - 4x(1-2x)q^\mu q^\nu P^2 \\
&\quad + 2x^2(1-2x)^2q^2q^\mu q^\nu + (1-4x)^2q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&\ \ - 2P^2P^\mu P^\nu + 2x(2-x)q^2P^\mu P^\nu - 2x(x+1)q^\mu q^\nu P^2 \\
&\quad + 2x^2(2-x)(x+1)q^2q^\mu q^\nu + 2(1-x)(2x+1)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&= -g^{\mu\nu}\left[(P^2)^2 + 4(x-1)^2q_\rho q_\sigma P^\rho P^\sigma + 2x(x-2)q^2P^2 + x^2(x-2)^2q^4\right] \\
&\ \ - P^2P^\mu P^\nu - (x+1)^2q^2P^\mu P^\nu - x^2q^\mu q^\nu P^2 \\
&\quad - x^2(x+1)^2q^2q^\mu q^\nu - 2x(x+1)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu)
\end{aligned}
$$

$$
\begin{aligned}
&-4P^2P^\mu P^\nu - 4x^2q^2P^\mu P^\nu - (1-2x)^2q^\mu q^\nu P^2 \\
&\quad - x^2(1-2x)^2q^2q^\mu q^\nu + 4x(1-2x)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&+ 8P^2P^\mu P^\nu + 4x(2x-1)q^2P^\mu P^\nu - 4x(1-2x)q^\mu q^\nu P^2 \\
&\quad + 2x^2(1-2x)^2q^2q^\mu q^\nu + (1-4x)^2q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
&- 2P^2P^\mu P^\nu + 2x(2-x)q^2P^\mu P^\nu - 2x(x+1)q^\mu q^\nu P^2 \\
&\quad + 2x^2(2-x)(x+1)q^2q^\mu q^\nu + 2(1-x)(2x+1)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
=& -g^{\mu\nu}\left[(P^2)^2 + 4(x-1)^2q_\rho q_\sigma P^\rho P^\sigma + 2x(x-2)q^2P^2 + x^2(x-2)^2q^4\right] \\
&+ P^2P^\mu P^\nu + (x^2-2x-1)q^2P^\mu P^\nu + (x^2-2x-1)q^\mu q^\nu P^2 \\
&\quad + x^2(x-2)^2q^2q^\mu q^\nu + (2x^2-4x+3)q_\rho P^\rho(P^\mu q^\nu + q^\mu P^\nu) \\
\to& -g^{\mu\nu}\left[(P^2)^2 + 4(x-1)^2q_\rho q_\sigma \frac{g^{\rho\sigma}}{d}P^2 + 2x(x-2)q^2P^2 + x^2(x-2)^2q^4\right] \\
&+ P^2P^\mu P^\nu + (x^2-2x-1)q^2\frac{g^{\mu\nu}}{d}P^2 + (x^2-2x-1)q^\mu q^\nu P^2 \\
&\quad + x^2(x-2)^2q^2q^\mu q^\nu + (2x^2-4x+3)q_\rho\left(q^\nu\frac{g^{\mu\rho}}{d}P^2 + q^\mu\frac{g^{\nu\rho}}{d}P^2\right) \\
=& -g^{\mu\nu}\left[(P^2)^2 + \left\{\frac{4}{d}(x-1)^2 + 2x(x-2)\right\}q^2P^2 + x^2(x-2)^2q^4\right] \\
&+ P^2P^\mu P^\nu + (x^2-2x-1)q^2\frac{g^{\mu\nu}}{d}P^2 + (x^2-2x-1)q^\mu q^\nu P^2 \\
&\quad + x^2(x-2)^2q^2q^\mu q^\nu + \frac{2}{d}(2x^2-4x+3)q^\mu q^\nu P^2 \\
=& P^2P^\mu P^\nu - g^{\mu\nu}(P^2)^2 - g^{\mu\nu}\left\{\frac{1}{d}(3x^2-6x+5) + 2x(x-2)\right\}q^2P^2 \\
&+ \left\{\frac{2}{d}(2x^2-4x+3) + (x^2-2x-1)\right\}q^\mu q^\nu P^2 - x^2(x-2)^2q^2(g^{\mu\nu}q^2 - q^\mu q^\nu).
\end{aligned}
$$

最後の項は積分すれば有限値

$$
2g^2(1-\xi)C_2(G)\delta^{ab}(g^{\mu\nu}q^2 - q^\mu q^\nu)\int_0^1 dx\,(1-x)\int\frac{d^dP}{(2\pi)^d}\frac{-x^2(x-2)^2q^2}{[P^2-\Delta]^3} \tag{16.5.2}
$$

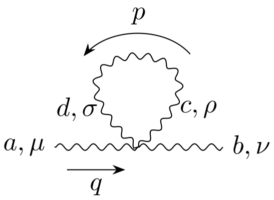
になるので，これ以降は考えない．

以上から (16.62) に加わる発散項は

$$
\begin{aligned}
&2g^2(1-\xi)C_2(G)\delta^{ab}\int_0^1 dx\,(1-x)\int\frac{d^dP}{(2\pi)^d}\frac{1}{[P^2-\Delta]^3} \\
&\quad\times\left[P^2P^\mu P^\nu - g^{\mu\nu}(P^2)^2 - g^{\mu\nu}\left\{\frac{1}{d}(3x^2-6x+5) + 2x(x-2)\right\}q^2P^2\right. \\
&\qquad\left. + \left\{\frac{2}{d}(2x^2-4x+3) + (x^2-2x-1)\right\}q^\mu q^\nu P^2\right]
\end{aligned} \tag{16.5.3}
$$

(16.65) の修正

(16.63) は



$$
\begin{aligned}
&= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{-i}{p^2} \left[g_{\rho\sigma} - (1-\xi) \frac{p_\rho p_\sigma}{p^2} \right] \delta^{cd} (-ig^2) \\
&\quad \times \left[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right] \\
&= (16.65) + \frac{g^2}{2} (1-\xi) \int \frac{d^d p}{(2\pi)^d} \frac{p_\rho p_\sigma}{p^4} \delta^{cd} \\
&\quad \times \left[f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \right] \\
&= (16.65) + \frac{g^2}{2} (1-\xi) \int \frac{d^d p}{(2\pi)^d} \frac{p_\rho p_\sigma}{p^4} f^{ace} f^{bce} \left[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} \right] \\
&= (16.65) + g^2 (1-\xi) C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4} \left[g^{\mu\nu} p^2 - p^\mu p^\nu \right] \\
&= (16.65) + g^2 (1-\xi) C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{g^{\mu\nu} p^2}{p^4} - g^2 (1-\xi) C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^4}.
\end{aligned}
$$

第 2 項の積分は

$$
\begin{aligned}
&\int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^4} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^4} \frac{(p+q)^2}{(p+q)^2} \\
&= 2 \int_0^1 dx\, (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{p^2 (p+q)^2}{[P^2 - \Delta]^3} \\
&= 2 \int_0^1 dx\, (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{(P - xq)^2 (P + (1-x)q)^2}{[P^2 - \Delta]^3} \\
&= 2 \int_0^1 dx\, (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
&\quad \times \left[P^2 - 2xP \cdot q + x^2 q^2 \right] \left[P^2 + 2(1-x) P \cdot q + (1-x)^2 q^2 \right] \\
&\to 2 \int_0^1 dx\, (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
&\quad \times \left[(P^2)^2 + (2x^2 - 2x + 1) q^2 P^2 - 4x(1-x) q_\rho q_\sigma P^\rho P^\sigma + x^2 (1-x)^2 q^4 \right] \\
&\to 2 \int_0^1 dx\, (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
&\quad \times \left[(P^2)^2 + (2x^2 - 2x + 1) q^2 P^2 - 4x(1-x) q_\rho q_\sigma \frac{g^{\rho\sigma}}{d} P^2 + x^2 (1-x)^2 q^4 \right] \\
&\to 2 \int_0^1 dx\, (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3}
\end{aligned}
$$

$$\times \left[(P^2)^2 + \left\{ (2x^2 - 2x + 1) - \frac{4}{d}x(1-x) \right\} q^2 P^2 + x^2(1-x)^2 q^4 \right].$$

第 3 項の積分は

$$\begin{aligned}
&\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^4} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^4} \frac{(p+q)^2}{(p+q)^2} \\
&= 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu (p+q)^2}{[P^2-\Delta]^3} \\
&= 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{(P-xq)^\mu (P-xq)^\nu (P+(1-x)q)^2}{[P^2-\Delta]^3} \\
&= 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[P^\mu P^\nu - x(P^\mu q^\nu + q^\mu P^\nu) + x^2 q^\mu q^\nu \right] \left[P^2 + 2(1-x) q \cdot P + (1-x)^2 q^2 \right] \\
&\to 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[P^2 P^\mu P^\nu + (1-x)^2 q^2 P^\mu P^\nu + x^2 q^\mu q^\nu P^2 \right. \\
&\quad\quad \left. + x^2(1-x)^2 q^\mu q^\nu q^2 - 2x(1-x) q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \right] \\
&\to 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[P^2 P^\mu P^\nu + (1-x)^2 q^2 \frac{g^{\mu\nu}}{d} P^2 + x^2 q^\mu q^\nu P^2 \right. \\
&\quad\quad \left. + x^2(1-x)^2 q^\mu q^\nu q^2 - 2x(1-x) q_\rho \left(q^\nu \frac{g^{\mu\rho}}{d} P^2 + q^\mu \frac{g^{\nu\rho}}{d} P^2 \right) \right] \\
&= 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[P^2 P^\mu P^\nu + (1-x)^2 q^2 \frac{g^{\mu\nu}}{d} P^2 + x^2 q^\mu q^\nu P^2 + x^2(1-x)^2 q^\mu q^\nu q^2 - \frac{4}{d} x(1-x) q^\mu q^\nu P^2 \right] \\
&\to 2\int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[P^2 P^\mu P^\nu + g^{\mu\nu} \frac{1}{d}(1-x)^2 q^2 P^2 + \left\{ x^2 - \frac{4}{d} x(1-x) \right\} q^\mu q^\nu P^2 + x^2(1-x)^2 q^\mu q^\nu q^2 \right].
\end{aligned}$$

以上から (16.65) に加わる項は

$$\begin{aligned}
&2g^2(1-\xi) C_2(G) \delta^{ab} \int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[g^{\mu\nu}(P^2)^2 + g^{\mu\nu} \left\{ (2x^2 - 2x + 1) - \frac{4}{d}x(1-x) \right\} q^2 P^2 + g^{\mu\nu} x^2(1-x)^2 q^4 \right] \\
&- 2g^2(1-\xi) C_2(G) \delta^{ab} \int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3} \\
&\quad \times \left[P^2 P^\mu P^\nu + g^{\mu\nu} \frac{1}{d}(1-x)^2 q^2 P^2 + \left\{ x^2 - \frac{4}{d} x(1-x) \right\} q^\mu q^\nu P^2 + x^2(1-x)^2 q^\mu q^\nu q^2 \right] \\
&= 2g^2(1-\xi) C_2(G) \delta^{ab} \int_0^1 dx\,(1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2-\Delta]^3}
\end{aligned}$$

$$
\times\left[-P^2P^\mu P^\nu+g^{\mu\nu}(P^2)^2+g^{\mu\nu}\left\{(2x^2-2x+1)+\frac{1}{d}(3x^2-2x-1)\right\}q^2P^2\right.
$$
$$
\left.-\left\{x^2-\frac{4}{d}x(1-x)\right\}q^\mu q^\nu P^2+x^2(1-x)^2q^2(q^2g^{\mu\nu}-q^\mu q^\nu)\right].
$$

最後の項

$$
2g^2(1-\xi)C_2(G)\delta^{ab}(q^2g^{\mu\nu}-q^\mu q^\nu)\int_0^1 dx\,(1-x)\int\frac{d^dp}{(2\pi)^d}\frac{x^2(1-x)^2q^2}{[P^2-\Delta]^3} \tag{16.5.4}
$$

は有限なので，これ以降考えない．以上から，(16.65) に加わる発散項は

$$
\begin{aligned}
&2g^2(1-\xi)C_2(G)\delta^{ab}\int_0^1 dx\,(1-x)\int\frac{d^dp}{(2\pi)^d}\frac{1}{[P^2-\Delta]^3}\\
&\quad\times\left[-P^2P^\mu P^\nu+g^{\mu\nu}(P^2)^2+g^{\mu\nu}\left\{(2x^2-2x+1)+\frac{1}{d}(3x^2-2x-1)\right\}q^2P^2\right.\\
&\qquad\left.-\left\{x^2-\frac{4}{d}x(1-x)\right\}q^\mu q^\nu P^2\right].
\end{aligned} \tag{16.5.5}
$$

[16.5.3] と [16.5.5] を足して，ゲージによる修正（のうち発散する部分）は

$$
\begin{aligned}
&2g^2(1-\xi)C_2(G)\delta^{ab}\int_0^1 dx\,(1-x)\int\frac{d^dP}{(2\pi)^d}\frac{1}{[P^2-\Delta]^3}\\
&\quad\times\left[P^2P^\mu P^\nu-g^{\mu\nu}(P^2)^2-g^{\mu\nu}\left\{\frac{1}{d}(3x^2-6x+5)+2x(x-2)\right\}q^2P^2\right.\\
&\qquad\qquad\left.+\left\{\frac{2}{d}(2x^2-4x+3)+(x^2-2x-1)\right\}q^\mu q^\nu P^2\right]\\
&+2g^2(1-\xi)C_2(G)\delta^{ab}\int_0^1 dx\,(1-x)\int\frac{d^dp}{(2\pi)^d}\frac{1}{[P^2-\Delta]^3}\\
&\quad\times\left[-P^2P^\mu P^\nu+g^{\mu\nu}(P^2)^2+g^{\mu\nu}\left\{(2x^2-2x+1)+\frac{1}{d}(3x^2-2x-1)\right\}q^2P^2\right.\\
&\qquad\qquad\left.-\left\{x^2-\frac{4}{d}x(1-x)\right\}q^\mu q^\nu P^2\right]\\
&=2g^2(1-\xi)C_2(G)\delta^{ab}\int_0^1 dx\,(1-x)\int\frac{d^dP}{(2\pi)^d}\frac{1}{[P^2-\Delta]^3}\\
&\quad\times\left[g^{\mu\nu}\left\{(2x+1)+\frac{2}{d}(2x-3)\right\}q^2P^2-\left\{(2x+1)+\frac{2}{d}(2x-3)\right\}q^\mu q^\nu P^2\right]\\
&=2g^2(1-\xi)C_2(G)\delta^{ab}(q^2g^{\mu\nu}-q^\mu q^\nu)\int_0^1 dx\,(1-x)\left\{(2x+1)+\frac{2}{d}(2x-3)\right\}\\
&\quad\times\int\frac{d^dP}{(2\pi)^d}\frac{P^2}{[P^2-\Delta]^3}\\
&=2g^2(1-\xi)C_2(G)\delta^{ab}(q^2g^{\mu\nu}-q^\mu q^\nu)\int_0^1 dx\,(1-x)\left\{(2x+1)+\frac{2}{d}(2x-3)\right\}\\
&\quad\times\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-d/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-d/2}\\
&=\frac{2ig^2}{(4\pi)^{d/2}}(1-\xi)C_2(G)\delta^{ab}(q^2g^{\mu\nu}-q^\mu q^\nu)\\
&\quad\times\int_0^1 dx\,(1-x)\left\{(2x+1)+\frac{2}{d}(2x-3)\right\}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}
\end{aligned}
$$

$$\approx i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left[\frac{-g^2}{(4\pi)^2}\frac{-1+\xi}{2}C_2(G)\Gamma(2-d/2)\right].$$

(16.71) と比べて

$$-\frac{5}{3} \mapsto -\frac{5}{3} + \frac{-1+\xi}{2} = -\frac{13}{6} + \frac{\xi}{2}$$

とすれば良いことが分かる．

(16.88)

(10.36) の直前で定義したように，

$$\psi_0 = \sqrt{Z_2}\psi, \quad A_0^\mu = \sqrt{Z_3}A^\mu.$$

(16.86) の第 2 項は

$$\bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 = Z_2\bar{\psi}(i\partial\!\!\!/ - m_0)\psi = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\psi}(i\delta_2\partial\!\!\!/ - \delta_m)\psi.$$

右辺の第 1 項は (16.34)，第 2 項は (16.87) に含まれる．他も同様．

16.6 Asymptotic Freedom: The Background Field Method

(16.99)

ラグランジアン (16.98) が不変なことを確かめる．随伴表現 $(t^b)_{ac} = if^{abc}$ を使う．

$$\tilde{A}_\mu^a = A_\mu^a + \mathcal{A}_\mu^a, \quad \tilde{D}_\mu^{ac} = D_\mu^{ac} + f^{abc}\mathcal{A}_\mu^b = \partial_\mu\delta^{ac} + f^{abc}\tilde{A}_\mu^b$$

とする．$\tilde{A}_\mu^a$ の変換は

$$\begin{aligned}
\tilde{A}_\mu^a &\to A_\mu^a + \mathcal{A}_\mu^a + (D_\mu\beta)^a - f^{abc}\beta^b\mathcal{A}_\mu^c \\
&= A_\mu^a + \mathcal{A}_\mu^a + D_\mu^{ac}\beta^c - f^{acb}\beta^c\mathcal{A}_\mu^b \\
&= A_\mu^a + \mathcal{A}_\mu^a + (\partial_\mu\delta^{ac} + f^{abc}A_\mu^b)\beta^c + f^{abc}\beta^c\mathcal{A}_\mu^b \\
&= A_\mu^a + \mathcal{A}_\mu^a + (\partial_\mu\delta^{ac} + f^{abc}A_\mu^b + f^{abc}\mathcal{A}_\mu^b)\beta^c \\
&= \tilde{A}_\mu^a + (\partial_\mu\delta^{ac} + f^{abc}\tilde{A}_\mu^b)\beta^c \\
&= \tilde{A}_\mu^a + \tilde{D}_\mu^{ac}\beta^c \\
&= \tilde{A}_\mu^a + (\tilde{D}_\mu\beta)^a \\
&= \tilde{A}_\mu^a + \partial_\mu\beta^a + f^{abc}\tilde{A}_\mu^b\beta^c.
\end{aligned}$$

fermion

フェルミオンの変換は

$$\psi^a \to \psi^a + i\beta^b(t^b\psi)_a = \psi^a + i\beta^b(t^b)_{ac}\psi^c = \psi^a - f^{abc}\beta^b\psi^c.$$

(15.30) と同様にフェルミオンの共変微分はフェルミオンと同じ変換になる．実際，

$$(\tilde{D}_\mu\psi)^a = \tilde{D}_\mu^{ac}\psi^c$$

$$
\begin{aligned}
&= (\partial_\mu \delta^{ac} + f^{abc}\tilde{A}^b_\mu)\psi^c \\
&= \partial_\mu \psi^a + f^{abc}\tilde{A}^b_\mu \psi^c \\
&\to \partial_\mu(\psi^a - f^{abc}\beta^b\psi^c) + f^{abc}(\tilde{A}^b_\mu + \partial_\mu \beta^b + f^{bde}\tilde{A}^d_\mu \beta^e)(\psi^c - f^{cfg}\beta^f\psi^g) \\
&\approx \partial_\mu \psi^a - f^{abc}(\partial_\mu \beta^b)\psi^c - f^{abc}\beta^b\partial_\mu\psi^c \\
&\quad + f^{abc}\left[\tilde{A}^b_\mu\psi^c + (\partial_\mu\beta^b)\psi^c + f^{bde}\tilde{A}^d_\mu\beta^e\psi^c - \tilde{A}^b_\mu f^{cfg}\beta^f\psi^g\right] \\
&= \partial_\mu\psi^a - f^{abc}\beta^b\partial_\mu\psi^c + f^{abc}\left[\tilde{A}^b_\mu\psi^c + f^{bde}\tilde{A}^d_\mu\beta^e\psi^c - \tilde{A}^b_\mu f^{cfg}\beta^f\psi^g\right] \\
&= \partial_\mu\psi^a + f^{abc}\tilde{A}^b_\mu\psi^c - f^{abc}\beta^b\partial_\mu\psi^c + f^{abc}f^{bde}\tilde{A}^d_\mu\beta^e\psi^c - f^{abc}f^{cfg}\tilde{A}^b_\mu\beta^f\psi^g \\
&= (\tilde{D}_\mu\psi)^a - f^{abc}\beta^b\partial_\mu\psi^c + f^{abc}f^{bde}\tilde{A}^d_\mu\beta^e\psi^c - f^{adb}f^{bec}\tilde{A}^d_\mu\beta^e\psi^c \\
&= (\tilde{D}_\mu\psi)^a - f^{abc}\beta^b\partial_\mu\psi^c + (f^{abc}f^{bde} + f^{ebc}f^{adb})\tilde{A}^d_\mu\beta^e\psi^c.
\end{aligned}
$$

Jacobi 恒等式 (15.70) から

$$
\begin{aligned}
&= (\tilde{D}_\mu\psi)^a - f^{abc}\beta^b\partial_\mu\psi^c - f^{dbc}f^{eab}\tilde{A}^d_\mu\beta^e\psi^c \\
&= (\tilde{D}_\mu\psi)^a - f^{abc}\beta^b\partial_\mu\psi^c - f^{cde}f^{abc}\tilde{A}^d_\mu\beta^b\psi^e \\
&= (\tilde{D}_\mu\psi)^a - f^{abc}\beta^b(\partial_\mu\psi^c + f^{cde}\tilde{A}^d_\mu\psi^e) \\
&= (\tilde{D}_\mu\psi)^a - f^{abc}\beta^b(\tilde{D}_\mu\psi)^c \\
&= (\delta^{ac} - f^{abc}\beta^b)(\tilde{D}_\mu\psi)^c.
\end{aligned}
$$

(13.98) のフェルミオンの項は

$$
\begin{aligned}
\bar{\psi}^a(i\not{D}^{ac} + \mathcal{A}^b_\mu\gamma^\mu(t^b)_{ac})\psi^c &= \bar{\psi}^a(i\not{D}^{ac} + if^{abc}\mathcal{A}^b_\mu\gamma^\mu)\psi^c \\
&= \bar{\psi}^a(iD^{ac}_\mu\gamma^\mu + if^{abc}\mathcal{A}^b_\mu\gamma^\mu)\psi^c \\
&= \bar{\psi}^a i(D^{ac}_\mu + f^{abc}\mathcal{A}^b_\mu)\gamma^\mu\psi^c \\
&= \bar{\psi}^a i\gamma^\mu\tilde{D}^{ac}_\mu\psi^c \\
&= \bar{\psi}^a i\gamma^\mu(\tilde{D}_\mu\psi)^a \\
&\to \bar{\psi}^d(\delta^{ad} - f^{abd}\beta^b)(\delta^{ac} - f^{abc}\beta^b)i\gamma^\mu(\tilde{D}_\mu\psi)_c \\
&\approx \bar{\psi}^d(\delta^{cd} - f^{cbd}\beta^b - f^{dbc}\beta^b)i\gamma^\mu(\tilde{D}_\mu\psi)_c \\
&= \bar{\psi}^d\delta^{cd}i\gamma^\mu(\tilde{D}_\mu\psi)_c \\
&= \bar{\psi}^c i\gamma^\mu(\tilde{D}_\mu\psi)_c
\end{aligned}
$$

なので不変.

ghost

ゴーストの変換はフェルミオンと同じ：

$$
c^a \to c^a - f^{abc}\beta^b c^c, \quad (\tilde{D}_\mu c)^a \to (\tilde{D}_\mu c)^a - f^{abc}\beta^b(\tilde{D}_\mu c)^c
$$

なので

$$
\begin{aligned}
&(D^2)^{ac}c^c + (D^\mu)^{ab}f^{bdc}\mathcal{A}^d_\mu c^c \\
&= (D^\mu)^{ab}D^{bc}_\mu c^c + (D^\mu)^{ab}f^{bdc}\mathcal{A}^d_\mu c^c \\
&= (D^\mu)^{ab}(D^{bc}_\mu + f^{bdc}\mathcal{A}^d_\mu)c^c
\end{aligned}
$$

$$
\begin{aligned}
&= (D^\mu)^{ab}\tilde{D}_\mu^{bc}c^c \\
&= (\partial_\mu\delta^{ab} + f^{adb}A_\mu^d)(\tilde{D}^\mu c)^b \\
&\to \left[\partial_\mu\delta^{ab} + f^{adb}(A_\mu^d + \partial_\mu\beta^d + f^{def}A_\mu^e\beta^f)\right](\delta^{bg} - f^{bhg}\beta^h)(\tilde{D}^\mu c)^g \\
&= \left[\partial_\mu\delta^{ab} + f^{adb}A_\mu^d + f^{adb}(\partial_\mu\beta^d + f^{def}A_\mu^e\beta^f)\right](\delta^{bg} - f^{bhg}\beta^h)(\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ab} + f^{adb}(\partial_\mu\beta^d) + f^{adb}f^{def}A_\mu^e\beta^f\right](\delta^{bg} - f^{bhg}\beta^h)(\tilde{D}^\mu c)^g \\
&\approx \left[D_\mu^{ag} + f^{adg}(\partial_\mu\beta^d) + f^{adg}f^{def}A_\mu^e\beta^f - f^{bhg}(D_\mu^{ab})\beta^h\right](\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ag} + f^{adg}(\partial_\mu\beta^d) + f^{adg}f^{def}A_\mu^e\beta^f - f^{bhg}(\partial_\mu\delta^{ab} + f^{acb}A_\mu^c)\beta^h\right](\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ag} + f^{adg}(\partial_\mu\beta^d) + f^{adg}f^{def}A_\mu^e\beta^f - f^{ahg}(\partial_\mu)\beta^h - f^{bhg}f^{acb}A_\mu^c\beta^h\right](\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ag} + f^{adg}(\partial_\mu\beta^d) + f^{adg}f^{def}A_\mu^e\beta^f - f^{ahg}(\partial_\mu\beta^h) - f^{ahg}\beta^h\partial_\mu - f^{bhg}f^{acb}A_\mu^c\beta^h\right](\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ag} + f^{abg}f^{bef}A_\mu^e\beta^f - f^{ahg}\beta^h\partial_\mu - f^{bfg}f^{aeb}A_\mu^e\beta^f\right](\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ag} + (f^{abg}f^{bef} + f^{fbg}f^{aeb})A_\mu^e\beta^f - f^{ahg}\beta^h\partial_\mu\right](\tilde{D}^\mu c)^g \\
&= \left[D_\mu^{ag} - f^{ebg}f^{fab}A_\mu^e\beta^f - f^{ahg}\beta^h\partial_\mu\right](\tilde{D}^\mu c)^g.
\end{aligned}
$$

(16.98) のゴーストの項は

$$
\begin{aligned}
&\bar{c}^a\left[(D^2)^{ac}c^c + (D^\mu)^{ab}f^{bdc}\mathcal{A}_\mu^d c^c\right] \\
&\to \bar{c}^c(\delta^{ac} - f^{adc}\beta^d)\left[D_\mu^{ag} - f^{ebg}f^{fab}A_\mu^e\beta^f - f^{ahg}\beta^h\partial_\mu\right](\tilde{D}^\mu c)^g \\
&\approx \bar{c}^c\left(D_\mu^{cg} - f^{ebg}f^{fcb}A_\mu^e\beta^f - f^{chg}\beta^h\partial_\mu - f^{adc}\beta^d D_\mu^{ag}\right)(\tilde{D}^\mu c)^g \\
&= \bar{c}^c\left[D_\mu^{cg} - f^{ebg}f^{fcb}A_\mu^e\beta^f - f^{cdg}\beta^d\partial_\mu - f^{adc}\beta^d(\partial_\mu\delta^{ag} + f^{abg}A_\mu^b)\right](\tilde{D}^\mu c)^g \\
&= \bar{c}^c\left[D_\mu^{cg} - f^{ebg}f^{fcb}A_\mu^e\beta^f - f^{adc}f^{abg}A_\mu^b\beta^d\right](\tilde{D}^\mu c)^g \\
&= \bar{c}^c\left[D_\mu^{cg} - f^{ebg}f^{fcb}A_\mu^e\beta^f - f^{bfc}f^{beg}A_\mu^e\beta^f\right](\tilde{D}^\mu c)^g \\
&= \bar{c}^c D_\mu^{cg}(\tilde{D}^\mu c)^g
\end{aligned}
$$

なので不変.

boson

ボソンの変換はゴーストと同じ

$$
\mathcal{A}_\mu^a \to \mathcal{A}_\mu^a - f^{abc}\beta^b\mathcal{A}_\mu^c, \quad (D^\mu\mathcal{A}_\mu)^a \to (D^\mu\mathcal{A}_\mu)^a - f^{abc}\beta^b(D^\mu\mathcal{A}_\mu)^c
$$

なので，ゲージ依存の項 $(D^\mu\mathcal{A}_\mu)^a)^2$ も不変.

$$
\tilde{F}_{\mu\nu}^a = \partial_\mu\tilde{A}_\nu^a - \partial_\nu\tilde{A}_\mu^a + f^{abc}\tilde{A}_\mu^b\tilde{A}_\nu^c = F_{\mu\nu}^a + (D_\mu\mathcal{A}_\nu)^a - (D_\nu\mathcal{A}_\mu)^a + f^{abc}\mathcal{A}_\mu^b\mathcal{A}_\nu^c
$$

の変換は

$$
\begin{aligned}
\tilde{F}_{\mu\nu}^a &= \partial_\mu\tilde{A}_\nu^a - \partial_\nu\tilde{A}_\mu^a + f^{abc}\tilde{A}_\mu^b\tilde{A}_\nu^c \\
&\to \partial_\mu(\tilde{A}_\nu^a + \partial_\nu\beta^a + f^{abc}\tilde{A}_\nu^b\beta^c) - \partial_\nu(\tilde{A}_\mu^a + \partial_\mu\beta^a + f^{abc}\tilde{A}_\mu^b\beta^c) \\
&\quad + f^{abc}(\tilde{A}_\mu^b + \partial_\mu\beta^b + f^{bde}\tilde{A}_\mu^d\beta^e)(\tilde{A}_\nu^c + \partial_\nu\beta^c + f^{cfg}\tilde{A}_f^f\beta^g) \\
&\approx \partial_\mu\tilde{A}_\nu^a - \partial_\nu\tilde{A}_\mu^a + f^{abc}\beta^c(\partial_\mu\tilde{A}_\nu^b - \partial_\nu\tilde{A}_\mu^b) + f^{abc}(\tilde{A}_\nu^b\partial_\mu - \tilde{A}_\mu^b\partial_\nu)\beta^c \\
&\quad + f^{abc}\tilde{A}_\mu^b\tilde{A}_\nu^c + f^{abc}\tilde{A}_\mu^b(\partial_\nu\beta^c + f^{cfg}\tilde{A}_\nu^f\beta^g) + f^{abc}\tilde{A}_\nu^c(\partial_\mu\beta^b + f^{bde}\tilde{A}_\mu^d\beta^e) \\
&= \partial_\mu\tilde{A}_\nu^a - \partial_\nu\tilde{A}_\mu^a + f^{abc}\tilde{A}_\mu^b\tilde{A}_\nu^c
\end{aligned}
$$

$$
\begin{aligned}
&+ f^{abc}\beta^c(\partial_\mu \tilde{A}^b_\nu - \partial_\nu \tilde{A}^b_\mu) + f^{abc}f^{cfg}\tilde{A}^b_\mu \tilde{A}^f_\nu \beta^g + f^{abc}f^{bde}\tilde{A}^d_\mu \tilde{A}^c_\nu \beta^e \\
&= \tilde{F}^a_{\mu\nu} + f^{abc}\beta^c(\partial_\mu \tilde{A}^b_\nu - \partial_\nu \tilde{A}^b_\mu) + f^{abc}f^{cfg}\tilde{A}^b_\mu \tilde{A}^f_\nu \beta^g + f^{acf}f^{cbg}\tilde{A}^b_\mu \tilde{A}^f_\nu \beta^g \\
&= \tilde{F}^a_{\mu\nu} + f^{abc}\beta^c(\partial_\mu \tilde{A}^b_\nu - \partial_\nu \tilde{A}^b_\mu) + (f^{abc}f^{cfg} + f^{gbc}f^{afc})\tilde{A}^b_\mu \tilde{A}^f_\nu \beta^g \\
&= \tilde{F}^a_{\mu\nu} - f^{abc}\beta^b(\partial_\mu \tilde{A}^c_\nu - \partial_\nu \tilde{A}^c_\mu) - f^{fbc}f^{gac}\tilde{A}^b_\mu \tilde{A}^f_\nu \beta^g \\
&= \tilde{F}^a_{\mu\nu} - f^{abc}\beta^b(\partial_\mu \tilde{A}^c_\nu - \partial_\nu \tilde{A}^c_\mu + f^{cde}\tilde{A}^d_\mu \tilde{A}^e_\nu) \\
&= \tilde{F}^a_{\mu\nu} - f^{abc}\beta^b \tilde{F}^c_{\mu\nu}.
\end{aligned}
$$

(13.98) のボソンの項は

$$
\begin{aligned}
(\tilde{F}^a_{\mu\nu})^2 &\to (\tilde{F}^a_{\mu\nu} - f^{abc}\beta^b \tilde{F}^c_{\mu\nu})(\tilde{F}^{\mu\nu}_a - f^{ade}\beta^d \tilde{F}^{\mu\nu}_e) \\
&\approx (\tilde{F}^a_{\mu\nu})^2 - f^{abc}\beta^b \tilde{F}^c_{\mu\nu}\tilde{F}^{\mu\nu}_a - f^{ade}\beta^d \tilde{F}^a_{\mu\nu}\tilde{F}^{\mu\nu}_e \\
&= (\tilde{F}^a_{\mu\nu})^2 - f^{abc}\beta^b \tilde{F}^c_{\mu\nu}\tilde{F}^{\mu\nu}_a - f^{eba}\beta^b \tilde{F}^e_{\mu\nu}\tilde{F}^{\mu\nu}_a \\
&= (\tilde{F}^a_{\mu\nu})^2
\end{aligned}
$$

なので不変.

(16.122)

ボソン $\mathcal{A}^a_\mu$ の場合を考える．(16.109) で定義したように

$$
(\Delta_{G,1})^{\mu\nu}_{ac} = -(D^2)_{ac}g^{\mu\nu} + 2F^b_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^{\mu\nu}(t^b)_{ac}.
$$

これを $(a,\mu;c,\nu)$ を添字とする $4d(r)\times 4d(r)$ 行列と考える．

$(\Delta_{G,1})^{\mu\nu}_{ac}$ の固有値と固有関数（p でラベルする）を

$$
(\Delta_{G,1})^{\mu\nu}_{ac}(x)\phi^{c(p)}_\nu(x) = V_{(p)}(x)\phi^{a(p)}_\mu(x)
$$

とおけば,

$$
\det \Delta_{G,1} = \prod_x \det \Delta_{G,1}(x) = \prod_x \prod_p V_{(p)}(x).
$$

よって

$$
\begin{aligned}
\log\det \Delta_{G,1} &= \sum_x \sum_p \log V_{(p)}(x) = \sum_x \mathrm{Tr}\log \Delta_{G,1}(x) = \int d^dx\ \mathrm{Tr}\log\Delta_{G,1}(x) \\
&= \int d^dx\ (\log \Delta_{G,1}(x))^{\mu\mu}_{aa} \\
&\sim \int d^dx\ \left[\log\left\{1 + (-\partial^2)^{-1}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})})\right\}\right]^{\mu\mu}_{aa} \\
&\approx \int d^dx\ \left[(-\partial^2)^{-1}(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})}) + \cdots\right]^{\mu\mu}_{aa}.
\end{aligned}
$$

このうち $\Delta^{(1)}$ の 2 乗を含む項は

$$
\begin{aligned}
&-\frac{1}{2}\int d^dx\ \langle x|\left[(-\partial^2)^{-1}\Delta^{(1)}(-\partial^2)^{-1}\Delta^{(1)}\right]^{\mu\mu}_{cc}|x\rangle \\
&= -\frac{g^{\mu\nu}g_{\nu\mu}}{2}\int d^dx\ \langle x|(-\partial^2)^{-1}\Delta^{(1)}_{cd}(-\partial^2)^{-1}\Delta^{(1)}_{dc}|x\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}\Delta^{(1)}_{cd}(-\partial^2)^{-1}\Delta^{(1)}_{dc}e^{-ip\cdot x}\,|p\rangle \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}\Delta^{(1)}_{cd}(-\partial^2)^{-1}\left[2A^b_\nu i\partial^\nu + (i\partial^\nu A^b_\nu)\right](t^b)_{dc}e^{-ip\cdot x}\,|p\rangle \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}\Delta^{(1)}_{cd}(-\partial^2)^{-1}e^{-ik\cdot x}(2p+k)^\nu A^b_\nu(k)e^{-ip\cdot x}\,|p\rangle\,(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}\Delta^{(1)}_{cd}(-\partial^2)^{-1}e^{-i(p+k)\cdot x}\,|p\rangle\,(2p+k)^\nu A^b_\nu(k)(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}\Delta^{(1)}_{cd}\frac{1}{(p+k)^2}e^{-i(p+k)\cdot x}\,|p\rangle\,(2p+k)^\nu A^b_\nu(k)(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}\left[2A^a_\mu i\partial^\mu + (i\partial^\mu A^a_\mu)\right](t^a)_{cd}e^{-i(p+k)\cdot x}\,|p\rangle \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2}A^b_\nu(k)(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\int \frac{d^dq}{(2\pi)^d}\, \langle x|\,(-\partial^2)^{-1}e^{-iq\cdot x}A^a_\mu(q)e^{-i(p+k)\cdot x}\,|p\rangle \\
&\quad \times (2p+2k+q)^\mu\frac{(2p+k)^\nu}{(p+k)^2}A^b_\nu(k)(t^a)_{cd}(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\int \frac{d^dq}{(2\pi)^d}\, \langle x|\,e^{-i(p+k+q)\cdot x}\,|p\rangle\,\frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2}A^a_\mu(q)A^b_\nu(k)(t^a)_{cd}(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\int \frac{d^dq}{(2\pi)^d}\, \langle x|p\rangle\,e^{-i(p+k+q)\cdot x}\frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2}A^a_\mu(q)A^b_\nu(k)(t^a)_{cd}(t^b)_{dc} \\
&= -2\int d^dx \int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\int \frac{d^dq}{(2\pi)^d} e^{-i(k+q)\cdot x}\frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2}A^a_\mu(q)A^b_\nu(k)(t^a)_{cd}(t^b)_{dc} \\
&= -2\int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\int \frac{d^dq}{(2\pi)^d}(2\pi)^d\,\delta^{(d)}(k+q)\frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2}A^a_\mu(q)A^b_\nu(k)(t^a)_{cd}(t^b)_{dc} \\
&= -2\int \frac{d^dp}{(2\pi)^d}\int \frac{d^dk}{(2\pi)^d}\frac{(2p+k)^\mu}{p^2}\frac{(2p+k)^\nu}{(p+k)^2}A^a_\mu(-k)A^b_\nu(k)\,\mathrm{Tr}(t^at^b).
\end{aligned}
$$

$\Delta^{(\mathcal{J})}$ の 2 乗を含む項で，Lorentz 添字についてトレースを取ると $(\mathcal{J}^{\rho\sigma})^{\lambda\kappa}(\mathcal{J}^{\alpha\beta})_{\kappa\lambda} = \mathrm{Tr}[\mathcal{J}^{\rho\sigma}\mathcal{J}^{\alpha\beta}]$ を得る．

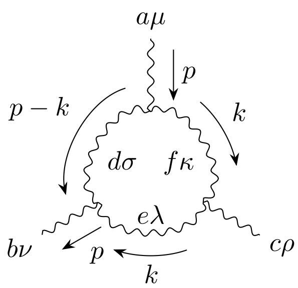
フェルミオンの場合は，Lorentz 添字 $\mu = 0,\ldots,3$ の代わりに Dirac スピノルの添字 $\alpha = 1,\ldots 4$ を考える．

Problems

Problem 16.3: Counterterm relations

(b)

3 ボソン頂点 3 つのダイアグラムを考える．



分母は $k^4(k-p)^2$ なので，$k=\ell+xp$ で，ℓ^2 の項のみが発散する．分子は[*1]

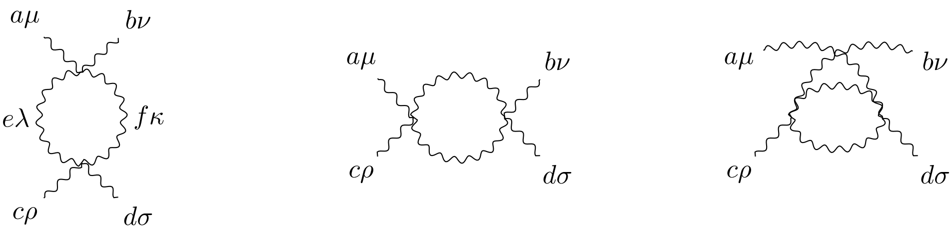
$$
\begin{aligned}
(\mathrm{Num}) = {} & 2k^2g^{\mu\nu}k^\rho + 4k^2g^{\mu\nu}p^\rho + 2k^2g^{\mu\rho}k^\nu - 3k^2g^{\mu\rho}p^\nu \\
& + 2k^2g^{\nu\rho}k^\mu - 3k^2g^{\nu\rho}p^\mu - 8(k\cdot p)g^{\mu\nu}p^\rho - 2(k\cdot p)g^{\mu\rho}k^\nu \\
& + 3(k\cdot p)g^{\mu\rho}p^\nu - 2(k\cdot p)g^{\nu\rho}k^\mu + 3(k\cdot p)g^{\nu\rho}p^\mu + 10p^2g^{\mu\nu}k^\rho \\
& - p^2g^{\mu\rho}k^\nu - p^2g^{\nu\rho}k^\mu + 18k^\mu k^\nu k^\rho - 9k^\mu k^\rho p^\nu \\
& + 3k^\mu p^\nu p^\rho - 9k^\nu k^\rho p^\mu + 3k^\nu p^\mu p^\rho - 6k^\rho p^\mu p^\nu \\
\sim {} & 2k^2g^{\mu\nu}k^\rho + 4k^2g^{\mu\nu}p^\rho + 2k^2g^{\mu\rho}k^\nu - 3k^2g^{\mu\rho}p^\nu \\
& + 2k^2g^{\nu\rho}k^\mu - 3k^2g^{\nu\rho}p^\mu - 2(k\cdot p)g^{\mu\rho}k^\nu \\
& - 2(k\cdot p)g^{\nu\rho}k^\mu \\
& + 18k^\mu k^\nu k^\rho - 9k^\mu k^\rho p^\nu \\
& - 9k^\nu k^\rho p^\mu \\
\sim {} & 2g^{\mu\nu}(\ell+xp)^2(\ell+xp)^\rho + 4g^{\mu\nu}(\ell+xp)^2p^\rho + 2g^{\mu\rho}(\ell+xp)^2(\ell+xp)^\nu - 3g^{\mu\rho}(\ell+xp)^2p^\nu \\
& + 2g^{\nu\rho}(\ell+xp)^2(\ell+xp)^\mu - 3g^{\nu\rho}(\ell+xp)^2p^\mu - 2g^{\mu\rho}p\cdot(\ell+xp)(\ell+xp)^\nu \\
& - 2g^{\nu\rho}p\cdot(\ell+xp)(\ell+xp)^\mu \\
& + 18(\ell+xp)^\mu(\ell+xp)^\nu(\ell+xp)^\rho - 9(\ell+xp)^\mu(\ell+xp)^\rho p^\nu \\
& - 9(\ell+xp)^\nu(\ell+xp)^\rho p^\mu \\
\sim {} & 2g^{\mu\nu}[xp^\rho\ell^2 + 2x(p\cdot\ell)\ell^\rho] + 4g^{\mu\nu}p^\rho\ell^2 + 2g^{\mu\rho}[xp^\nu\ell^2 + 2x(p\cdot\ell)\ell^\nu] - 3g^{\mu\rho}p^\nu\ell^2 \\
& + 2g^{\nu\rho}[xp^\mu\ell^2 + 2x(p\cdot\ell)\ell^\mu] - 3g^{\nu\rho}p^\mu\ell^2 - 2g^{\mu\rho}(p\cdot\ell)\ell^\nu - 2g^{\nu\rho}(p\cdot\ell)\ell^\mu \\
& + 18x(p^\mu\ell^\nu\ell^\rho + p^\nu\ell^\rho\ell^\mu + p^\rho\ell^\mu\ell^\nu) - 9p^\nu\ell^\mu\ell^\rho - 9p^\mu\ell^\nu\ell^\rho \\
\sim {} & 3xg^{\mu\nu}p^\rho\ell^2 + 4g^{\mu\nu}p^\rho\ell^2 + 3xg^{\mu\rho}p^\nu\ell^2 - 3g^{\mu\rho}p^\nu\ell^2 \\
& + 3xg^{\nu\rho}p^\mu\ell^2 - 3g^{\nu\rho}p^\mu\ell^2 - \frac{1}{2}g^{\mu\rho}p^\nu\ell^2 - \frac{1}{2}g^{\nu\rho}p^\mu\ell^2 \\
& + \frac{18x-9}{4}g^{\nu\rho}p^\mu\ell^2 + \frac{18x-9}{4}g^{\mu\rho}p^\nu\ell^2 + \frac{9}{2}xg^{\mu\nu}p^\rho\ell^2
\end{aligned}
$$

[*1] `./src/py/NonAbelian_1loop.ipynb`

$$= \frac{15x+8}{2} g^{\mu\nu} p^\rho \ell^2 + \frac{30x-23}{4} (g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu) \ell^2.$$

(c)

4 ボソン頂点 2 つのダイアグラムを考える．



1 つ目のダイアグラムは

$$\begin{aligned}
&[f_{abg} f_{efg} (-g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}) + f_{aeg} f_{bfg} (g_{\kappa\lambda} g_{\mu\nu} - g_{\mu\kappa} g_{\nu\lambda}) + f_{afg} f_{beg} (g_{\kappa\lambda} g_{\mu\nu} - g_{\mu\lambda} g_{\nu\kappa})] \\
&\quad \times [f_{cdh} f_{efh} (-g_{\rho\kappa} g_{\sigma\lambda} + g_{\rho\lambda} g_{\sigma\kappa}) + f_{ech} f_{fdh} (g_{\kappa\lambda} g_{\rho\sigma} - g_{\rho\kappa} g_{\sigma\lambda}) + f_{edh} f_{fch} (g_{\kappa\lambda} g_{\rho\sigma} - g_{\rho\lambda} g_{\sigma\kappa})] \\
&= 2 f^{abg} f^{cdh} f^{efg} f^{efh} g^{\mu\rho} g^{\nu\sigma} - 2 f^{abg} f^{cdh} f^{efg} f^{efh} g^{\mu\sigma} g^{\nu\rho} \\
&\quad + f^{abg} f^{ech} f^{efg} f^{fdh} g^{\mu\rho} g^{\nu\sigma} - f^{abg} f^{ech} f^{efg} f^{fdh} g^{\mu\sigma} g^{\nu\rho} \\
&\quad - f^{abg} f^{edh} f^{efg} f^{fch} g^{\mu\rho} g^{\nu\sigma} + f^{abg} f^{edh} f^{efg} f^{fch} g^{\mu\sigma} g^{\nu\rho} \\
&\quad + f^{aeg} f^{bfg} f^{cdh} f^{efh} g^{\mu\rho} g^{\nu\sigma} - f^{aeg} f^{bfg} f^{cdh} f^{efh} g^{\mu\sigma} g^{\nu\rho} \\
&\quad + 2 f^{aeg} f^{bfg} f^{ech} f^{fdh} g^{\mu\nu} g^{\rho\sigma} + f^{aeg} f^{bfg} f^{ech} f^{fdh} g^{\mu\rho} g^{\nu\sigma} \\
&\quad + 2 f^{aeg} f^{bfg} f^{edh} f^{fch} g^{\mu\nu} g^{\rho\sigma} + f^{aeg} f^{bfg} f^{edh} f^{fch} g^{\mu\sigma} g^{\nu\rho} \\
&\quad - f^{afg} f^{beg} f^{cdh} f^{efh} g^{\mu\rho} g^{\nu\sigma} + f^{afg} f^{beg} f^{cdh} f^{efh} g^{\mu\sigma} g^{\nu\rho} \\
&\quad + 2 f^{afg} f^{beg} f^{ech} f^{fdh} g^{\mu\nu} g^{\rho\sigma} + f^{afg} f^{beg} f^{ech} f^{fdh} g^{\mu\sigma} g^{\nu\rho} \\
&\quad + 2 f^{afg} f^{beg} f^{edh} f^{fch} g^{\mu\nu} g^{\rho\sigma} + f^{afg} f^{beg} f^{edh} f^{fch} g^{\mu\rho} g^{\nu\sigma} \\
&= 2 f^{abg} f^{cdh} f^{efg} f^{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + (f^{abg} f^{ech} f^{efg} f^{fdh} - f^{abg} f^{edh} f^{efg} f^{fch}) (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + (f^{aeg} f^{bfg} f^{cdh} f^{efh} - f^{afg} f^{beg} f^{cdh} f^{efh}) (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + (f^{aeg} f^{bfg} f^{ech} f^{fdh} + f^{afg} f^{beg} f^{edh} f^{fch}) (2 g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}) \\
&\quad + (f^{aeg} f^{bfg} f^{edh} f^{fch} + f^{afg} f^{beg} f^{ech} f^{fdh}) (2 g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
&= 2 f^{abg} f^{cdh} f^{efg} f^{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + 2 f^{abg} f^{ech} f^{efg} f^{fdh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + 2 f^{aeg} f^{bfg} f^{cdh} f^{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + 2 f^{aeg} f^{bfg} f^{ech} f^{fdh} (2 g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}) \\
&\quad + 2 f^{aeg} f^{bfg} f^{edh} f^{fch} (2 g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho})
\end{aligned}$$

1 行目は

$$2 f^{abg} f^{cdh} f^{efg} f^{efh} = 2 f^{abg} f^{cdh} C_2(G) \delta^{gh} = 2 f^{abg} f^{cdg} C_2(G).$$

(16.79) から

$$i \operatorname{Tr}(t_G^a t_G^b t_G^c) = f^{ade} f^{ebf} f^{fdc} = -\frac{1}{2} f^{abc} C_2(G) \tag{16.6.6}$$

なので，2 行目は

$$2f^{abg}f^{ech}f^{efg}f^{fdh} = -2f^{abg}f^{ceh}f^{hdf}f^{feg} = f^{abg}f^{cdg}C_2(G).$$

同様に 3 行目は

$$2f^{aeg}f^{bfg}f^{cdh}f^{efh} = -2f^{cdh}f^{aeg}f^{gbf}f^{feh} = f^{cdh}f^{abh}C_2(G).$$

以上から，

$$\begin{aligned}
&= 4C_2(G)f^{abg}f^{cdg}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2f^{aeg}f^{bfg}f^{ech}f^{fdh}(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) \\
&\quad + 2f^{aeg}f^{bfg}f^{edh}f^{fch}(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&= 4C_2(G)f^{abg}f^{cdg}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}).
\end{aligned}$$

随伴表現は

$$(t_G^{b\dagger})_{ac} = (t_G^b)^*_{ca} = (if_{cba})^* = -if_{cba} = if_{abc} = (t_G^b)_{ac}$$

なので t_G^a は Hermite 行列．さらに $t_G^a t_G^b t_G^c t_G^d$ は実数なので

$$\begin{aligned}
\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d) &= (t_G^a)_{ij}(t_G^b)_{jk}(t_G^c)_{kl}(t_G^d)_{li} = (t_G^a)^*_{ij}(t_G^b)^*_{jk}(t_G^c)^*_{kl}(t_G^d)^*_{li} \\
&= (t_G^a)_{ji}(t_G^b)_{kj}(t_G^c)_{lk}(t_G^d)_{il} \\
&= \mathrm{Tr}(t_G^a t_G^d t_G^c t_G^b).
\end{aligned} \tag{16.6.7}$$

[16.6.6] と併せて

$$\begin{aligned}
C_2(G)f^{abg}f^{cdg} &= -2i\,\mathrm{Tr}(t_G^a t_G^b t_G^g)f^{cdg} = -2\,\mathrm{Tr}(t_G^a t_G^b [t_G^c, t_G^d]) \\
&= -2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d) + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c).
\end{aligned} \tag{16.6.8}$$

よって，

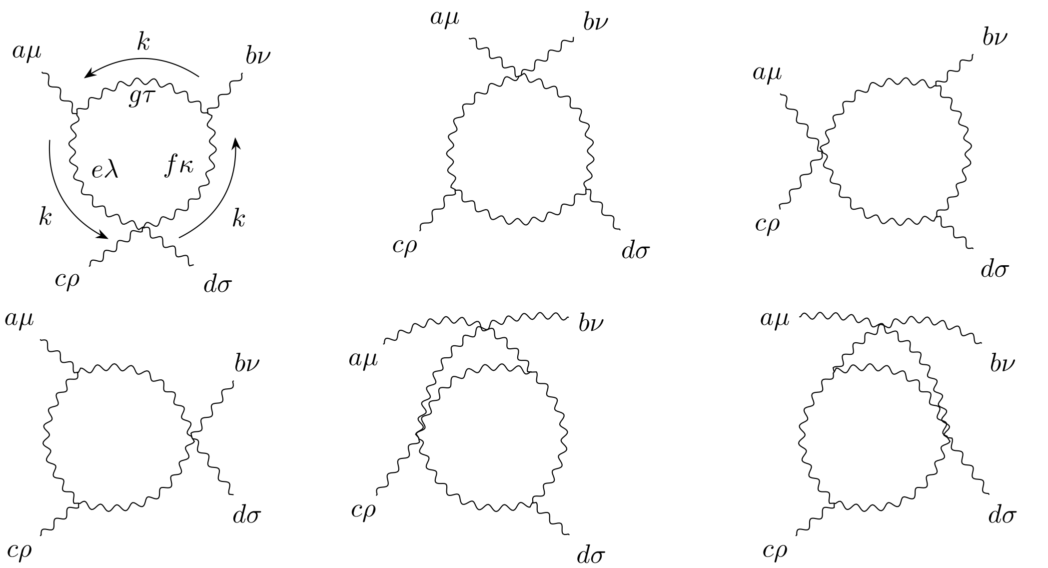
$$\begin{aligned}
&= 4[-2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d) + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)](g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\
&= 2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\nu}g^{\rho\sigma} + 5g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\sigma}g^{\nu\rho}).
\end{aligned}$$

残りのダイアグラムは $((b\nu \leftrightarrow c\rho))$，$((b\nu \leftrightarrow d\sigma))$ によって得られる．合計は

$$\begin{aligned}
&= 2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^c t_G^b t_G^d)(2g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\nu}g^{\rho\sigma} + 5g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\nu}g^{\rho\sigma}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\nu}g^{\rho\sigma} + 5g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^c t_G^b t_G^d)(2g^{\mu\sigma}g^{\nu\rho} + 5g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\nu}g^{\rho\sigma}) \\
&= 2\,\mathrm{Tr}(t_G^a t_G^b t_G^c t_G^d)[7g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 7g^{\mu\sigma}g^{\nu\rho}]
\end{aligned}$$

$$
\begin{aligned}
&+ 2\,\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)[7g^{\mu\nu}g^{\rho\sigma} + 7g^{\mu\rho}g^{\nu\sigma} - 8g^{\mu\sigma}g^{\nu\rho}] \\
&\quad + 2\,\mathrm{Tr}(t_G^a t_G^c t_G^b t_G^d)[-8g^{\mu\nu}g^{\rho\sigma} + 7g^{\mu\rho}g^{\nu\sigma} + 7g^{\mu\sigma}g^{\nu\rho}].
\end{aligned}
$$

4 ボソン頂点 1 つと 3 ボソン頂点 2 つのダイアグラムを考える．



1 つ目のダイアグラムは $f^{aeg}f^{bgf} \times (efcd)\times$

$$
\begin{aligned}
&(-2g_{\lambda\tau}k_\mu + g_{\mu\lambda}k_\tau + g_{\mu\tau}k_\lambda)(-2g_{\kappa\tau}k_\nu + g_{\nu\kappa}k_\tau + g_{\nu\tau}k_\kappa) \\
&= g_{\mu\lambda}g_{\nu\kappa}k^2 + 4g_{\kappa\lambda}k_\mu k_\nu - 2g_{\mu\kappa}k_\lambda k_\nu - g_{\mu\lambda}k_\kappa k_\nu + g_{\mu\nu}k_\kappa k_\lambda - g_{\nu\kappa}k_\lambda k_\mu - 2g_{\nu\lambda}k_\kappa k_\mu \\
&\sim g_{\mu\lambda}g_{\nu\kappa}k^2 + g_{\kappa\lambda}g_{\mu\nu}k^2 - \frac{1}{2}g_{\mu\kappa}g_{\nu\lambda}k^2 - \frac{1}{4}g_{\mu\lambda}g_{\nu\kappa}k^2 + \frac{1}{4}g_{\mu\nu}g_{\lambda\kappa}k^2 - \frac{1}{4}g_{\nu\kappa}g_{\mu\lambda}k^2 - \frac{1}{2}g_{\nu\lambda}g_{\mu\kappa}k^2 \\
&= \frac{5}{4}g_{\mu\nu}g_{\lambda\kappa}k^2 + \frac{1}{2}g_{\mu\lambda}g_{\nu\kappa}k^2 - g_{\mu\kappa}g_{\nu\lambda}k^2.
\end{aligned}
$$

従って，k^2 の係数は $f^{aeg}f^{bgf}/4\times$

$$
\begin{aligned}
&\left[f^{cdh}f^{efh}\left(-g^{\rho\kappa}g^{\sigma\lambda} + g^{\rho\lambda}g^{\sigma\kappa}\right) + f^{ech}f^{fdh}\left(g^{\kappa\lambda}g^{\rho\sigma} - g^{\rho\kappa}g^{\sigma\lambda}\right) + f^{edh}f^{fch}\left(g^{\kappa\lambda}g^{\rho\sigma} - g^{\rho\lambda}g^{\sigma\kappa}\right)\right] \\
&\quad \times \left(5g^{\mu\nu}g^{\lambda\kappa} + 2g^{\mu\lambda}g^{\nu\kappa} - 4g^{\mu\kappa}g^{\nu\lambda}\right) \\
&= 6f^{cdh}f^{efh}g^{\mu\rho}g^{\nu\sigma} - 6f^{cdh}f^{efh}g^{\mu\sigma}g^{\nu\rho} + 13f^{ech}f^{fdh}g^{\mu\nu}g^{\rho\sigma} + 4f^{ech}f^{fdh}g^{\mu\rho}g^{\nu\sigma} \\
&\quad - 2f^{ech}f^{fdh}g^{\mu\sigma}g^{\nu\rho} + 13f^{edh}f^{fch}g^{\mu\nu}g^{\rho\sigma} - 2f^{edh}f^{fch}g^{\mu\rho}g^{\nu\sigma} + 4f^{edh}f^{fch}g^{\mu\sigma}g^{\nu\rho} \\
&= 6f^{cdh}f^{efh}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + f^{ech}f^{fdh}(13g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}) \\
&\qquad + f^{edh}f^{fch}(13g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}).
\end{aligned}
$$

[16.6.6][16.6.8] を使えば

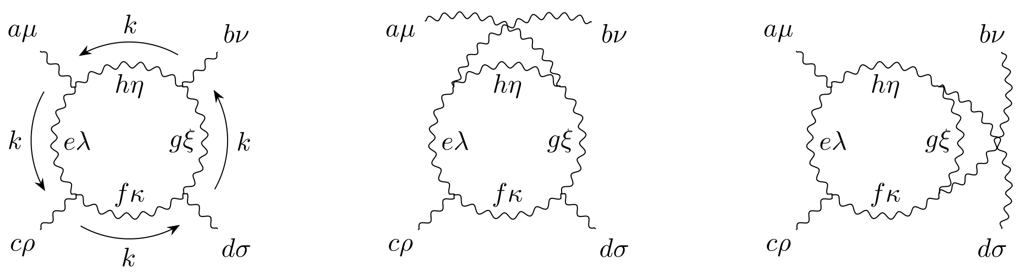
$$
\begin{aligned}
&\frac{3}{2}f^{aeg}f^{bgf}f^{cdh}f^{efh}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + \frac{1}{4}f^{aeg}f^{bgf}f^{ech}f^{fdh}(13g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}) \\
&\quad + \frac{1}{4}f^{aeg}f^{bgf}f^{edh}f^{fch}(8g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho})
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{3}{4} C_2(G) f^{abh} f^{cdh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad - \frac{1}{4} \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c)(13 g^{\mu\nu} g^{\rho\sigma} + 4 g^{\mu\rho} g^{\nu\sigma} - 2 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad - \frac{1}{4} \operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d)(13 g^{\mu\nu} g^{\rho\sigma} - 2 g^{\mu\rho} g^{\nu\sigma} + 4 g^{\mu\sigma} g^{\nu\rho}) \\
&= \frac{3}{2} \left[\operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d) - \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c) \right] (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&\quad - \frac{1}{4} \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c)(13 g^{\mu\nu} g^{\rho\sigma} + 4 g^{\mu\rho} g^{\nu\sigma} - 2 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad - \frac{1}{4} \operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d)(13 g^{\mu\nu} g^{\rho\sigma} - 2 g^{\mu\rho} g^{\nu\sigma} + 4 g^{\mu\sigma} g^{\nu\rho}) \\
&= \frac{1}{4} \operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d)(-13 g^{\mu\nu} g^{\rho\sigma} + 8 g^{\mu\rho} g^{\nu\sigma} - 10 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + \frac{1}{4} \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c)(-13 g^{\mu\nu} g^{\rho\sigma} - 10 g^{\mu\rho} g^{\nu\sigma} + 8 g^{\mu\sigma} g^{\nu\rho}).
\end{aligned}
$$

残りのダイアグラムは $((a\mu, b\nu \leftrightarrow c\rho, d\sigma))$，$((a\mu \leftrightarrow d\sigma))$，$((b\nu \leftrightarrow c\rho))$，$((b\nu \leftrightarrow d\sigma))$，$((a\mu \leftrightarrow c\rho))$ の置換によって得られる．[16.6.7] に注意して全て足せば（1, 2 個目及び 3, 4 個目及び 5, 6 個目はそれぞれ同じ値になる），

$$
\begin{aligned}
&\frac{1}{2} \operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d)(-13 g^{\mu\nu} g^{\rho\sigma} + 8 g^{\mu\rho} g^{\nu\sigma} - 10 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + \frac{1}{2} \operatorname{Tr}(t_G^a t_G^c t_G^b t_G^d)(-13 g^{\nu\sigma} g^{\mu\rho} + 8 g^{\rho\sigma} g^{\nu\mu} - 10 g^{\mu\sigma} g^{\nu\rho}) \\
&\qquad + \frac{1}{2} \operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d)(-13 g^{\mu\sigma} g^{\nu\rho} + 8 g^{\mu\rho} g^{\nu\sigma} - 10 g^{\mu\nu} g^{\rho\sigma}) \\
&+ \frac{1}{2} \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c)(-13 g^{\mu\nu} g^{\rho\sigma} - 10 g^{\mu\rho} g^{\nu\sigma} + 8 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + \frac{1}{2} \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c)(-13 g^{\nu\sigma} g^{\mu\rho} - 10 g^{\rho\sigma} g^{\mu\nu} + 8 g^{\mu\sigma} g^{\nu\rho}) \\
&\qquad + \frac{1}{2} \operatorname{Tr}(t_G^a t_G^c t_G^b t_G^d)(-13 g^{\mu\sigma} g^{\nu\rho} - 10 g^{\mu\rho} g^{\nu\sigma} + 8 g^{\mu\nu} g^{\rho\sigma}) \\
&= \operatorname{Tr}(t_G^a t_G^b t_G^c t_G^d)(-23 g^{\mu\nu} g^{\rho\sigma} + 16 g^{\mu\rho} g^{\nu\sigma} - 23 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c)(-23 g^{\mu\nu} g^{\rho\sigma} - 23 g^{\mu\rho} g^{\nu\sigma} + 16 g^{\mu\sigma} g^{\nu\rho}) \\
&\qquad + \operatorname{Tr}(t_G^a t_G^c t_G^b t_G^d)(16 g^{\mu\nu} g^{\rho\sigma} - 23 g^{\mu\rho} g^{\nu\sigma} - 23 g^{\mu\sigma} g^{\nu\rho}).
\end{aligned}
$$

3 ボソン頂点 4 つのダイアグラムを考える．



1 つ目のダイアグラムは $f^{aeh} f^{ecf} f^{gfd} f^{bhg} \times$

$$
\begin{aligned}
&\left(-2 g^{\lambda\eta} k^\mu + g^{\mu\eta} k^\lambda + g^{\mu\lambda} k^\eta\right) \left(-2 g^{\kappa\lambda} k^\rho + g^{\rho\kappa} k^\lambda + g^{\rho\lambda} k^\kappa\right) \\
&\quad \times \left(-2 g^{\kappa\xi} k^\sigma + g^{\sigma\kappa} k^\xi + g^{\sigma\xi} k^\kappa\right) \left(g^{\nu\eta} k^\xi + g^{\nu\xi} k^\eta - 2 g^{\xi\eta} k^\nu\right)
\end{aligned}
$$

$$= k^4 g^{\mu\nu} g^{\rho\sigma} + k^4 g^{\mu\rho} g^{\nu\sigma} + 3k^2 g^{\mu\nu} k^\rho k^\sigma + 3k^2 g^{\mu\rho} k^\nu k^\sigma + 3k^2 g^{\nu\sigma} k^\mu k^\rho + 3k^2 g^{\rho\sigma} k^\mu k^\nu + 34 k^\mu k^\nu k^\rho k^\sigma$$

となる．(A.47)(A.48) を参考にすれば

$$k^\mu k^\nu k^\rho k^\sigma \to \frac{1}{24}(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

として良いことが分かる．よって，k^4 の係数は $\mathrm{Tr}(t_G^a t_G^b t_G^d t_G^c)\times$

$$\begin{aligned}
&g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + \frac{3}{4} g^{\mu\nu} g^{\rho\sigma} + \frac{3}{4} k^2 g^{\mu\rho} g^{\nu\sigma} + \frac{3}{4} g^{\nu\sigma} g^{\mu\rho} + \frac{3}{4} g^{\rho\sigma} g^{\mu\nu} + \frac{17}{12}(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
&= \frac{47}{12} g^{\mu\nu} g^{\rho\sigma} + \frac{47}{12} g^{\mu\rho} g^{\nu\sigma} + \frac{17}{12} g^{\mu\sigma} g^{\nu\rho}.
\end{aligned}$$

残りのダイアグラムは $(\!(a\mu \leftrightarrow b\nu)\!)$，$(\!(b\nu \leftrightarrow d\sigma)\!)$ によって得られる．

Chapter 17

Quantum Chromodynamics

17.4 Hard-Scattering Processes in Hadron Collisions

(17.58)

Jacobian を計算するコード.

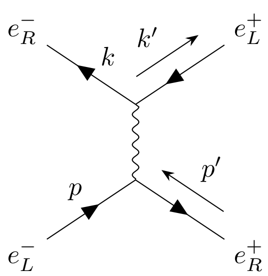
```
import sympy
p = sympy.Symbol('p')
y_3 = sympy.Symbol('y_3')
y_4 = sympy.Symbol('y_4')
y = (y_3-y_4)/2
Y = (y_3+y_4)/2

x_1 = 2 * p * sympy.cosh(y) * sympy.exp(Y)
x_2 = 2 * p * sympy.cosh(y) * sympy.exp(-Y)
t = -2 * p**2 * sympy.cosh(y) * sympy.exp(-y)

O = sympy.Matrix([x_1, x_2, t])
N = sympy.Matrix([y_3, y_4, p])
J = sympy.det(O.jacobian(N))
print(sympy.simplify(J))
```

(17.67)

$e_L^- e_R^+ \to e_R^- e_L^+$ の過程を考える.



Section 5.2 と同様に，射影演算子 $(1 \pm \gamma^5)/2$ を使えば

$$\begin{aligned} i\mathcal{M} &= ie^2 \bar{u}(k)\gamma^\mu \frac{1-\gamma^5}{2} v(k') \frac{1}{(p+p')^2} \bar{v}(p')\gamma_\mu \frac{1-\gamma^5}{2} u(p) \\ &= \frac{ie^2}{s} \bar{u}(k)\gamma^\mu \frac{1-\gamma^5}{2} v(k')\bar{v}(p')\gamma_\mu \frac{1-\gamma^5}{2} u(p) \end{aligned}$$

なので

$$\begin{aligned} |\mathcal{M}|^2 = \frac{e^4}{s^2} &\bar{u}(k)\gamma^\mu \frac{1-\gamma^5}{2} v(k')\bar{v}(k')\gamma^\nu \frac{1-\gamma^5}{2} u(k) \\ &\times \bar{v}(p')\gamma_\mu \frac{1-\gamma^5}{2} u(p)\bar{u}(p)\gamma_\nu \frac{1-\gamma^5}{2} v(p'). \end{aligned}$$

入射電子・陽電子に関してはスピンを平均，散乱電子・陽電子に関してはスピンを合計して，

$$\begin{aligned} \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{4s^2} \operatorname{Tr}\left(\not{k}\gamma^\mu \frac{1-\gamma^5}{2} \not{k}'\gamma^\nu \frac{1-\gamma^5}{2} \right) \operatorname{Tr}\left(\not{p}'\gamma_\mu \frac{1-\gamma^5}{2} \not{p}\gamma_\nu \frac{1-\gamma^5}{2} \right) \\ &= \frac{e^4}{4s^2} \operatorname{Tr}\left(\not{k}\gamma^\mu \not{k}'\gamma^\nu \frac{1-\gamma^5}{2} \right) \operatorname{Tr}\left(\not{p}'\gamma_\mu \not{p}\gamma_\nu \frac{1-\gamma^5}{2} \right) \\ &= \frac{e^4}{4s^2} E^2 (1-\cos\theta)^2 \\ &= \frac{e^4}{4} \left(\frac{t}{s} \right)^2 . \end{aligned}$$

トレースの計算は (5.23) の結果を利用した．(4.85) から

$$\frac{d\sigma}{d\cos\theta} = \frac{|\mathcal{M}|^2}{32\pi s} = \frac{e^2}{128\pi s} \left(\frac{t}{s} \right)^2 = \frac{\pi\alpha^2}{8s} \left(\frac{t}{s} \right)^2 .$$

さらに

$$t = -E^2(1-\cos\theta)^2 - E^2 \sin^2\theta = E^2(2\cos\theta - 2) = \frac{s}{2}(\cos\theta - 1)$$

なので

$$\frac{d\sigma}{dt} = \frac{2}{s}\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{4s^2} \left(\frac{t}{s} \right)^2 .$$

Problems

Problem 17.4: The gluon splitting function

$P_{g\leftarrow g}(z,)$ の規格化条件を求める．(17.39) と同様に

$$\int_0^1 dz\, z \left[\sum_f \left\{ f_f(z,Q) + f_{\bar{f}}(z,Q) \right\} + f_g(z,Q) \right] = 1.$$

(17.128) を積分して

$$\frac{d}{d\log Q} \int_0^1 dx\, x f_g(x,Q)$$

$$
\begin{aligned}
&= \frac{\alpha_s}{\pi}\int_0^1 dx\,x\int_x^1 \frac{dz}{z}\left[P_{gq}(z)\sum_f \left\{f_f(x/z)+f_{\bar f}(x/z)\right\}+P_{gg}(z)f_g(x/z)\right] \\
&= \frac{\alpha_s}{\pi}\int_0^1 ds\,s\int_0^1 dz\,z\left[P_{gq}(z)\sum_f \left\{f_f(s)+f_{\bar f}(s)\right\}+P_{gg}(z)f_g(s)\right]
\end{aligned}
$$

となる（$s=x/z$ とした）．他の式も同様にして

$$
\begin{aligned}
\frac{d}{d\log Q}\int_0^1 dx\,xf_f(x,Q) &= \frac{\alpha_s}{\pi}\int_0^1 ds\,s\int_0^1 dz\,z\left[P_{qq}(z)f_f(s)+P_{qg}(z)f_g(s)\right], \\
\frac{d}{d\log Q}\int_0^1 dx\,xf_{\bar f}(x,Q) &= \frac{\alpha_s}{\pi}\int_0^1 ds\,s\int_0^1 dz\,z\left[P_{qq}(z)f_{\bar f}(s)+P_{qg}(z)f_g(s)\right].
\end{aligned}
$$

以上から，

$$
\begin{aligned}
0 &= \frac{d}{d\log Q}\int_0^1 dx\,x\left[\sum_f \left\{f_f(x,Q)+f_{\bar f}(x,Q)\right\}+f_g(x,Q)\right] \\
&= \frac{\alpha_s}{\pi}\int_0^1 ds\,s\int_0^1 dz\,z\left[\{P_{gq}(z)+P_{qq}(z)\}\sum_f \left\{f_f(s)+f_{\bar f}(s)\right\}+\{2n_fP_{qg}(z)+P_{gg}(z)\}f_g(s)\right] \\
&= \frac{\alpha_s}{\pi}\int_0^1 ds\,s\left\{f_f(s)+f_{\bar f}(s)\right\}\int_0^1 dz\,z\{P_{gq}(z)+P_{qq}(z)\} \\
&\quad +\frac{\alpha_s}{\pi}\int_0^1 ds\,sf_g(s)\int_0^1 dz\,z\{2n_fP_{qg}(z)+P_{gg}(z)\} \\
&= -\frac{\alpha_s}{\pi}\int_0^1 ds\,sf_g(s)\int_0^1 dz\,z\{P_{gq}(z)+P_{qq}(z)\}+\frac{\alpha_s}{\pi}\int_0^1 ds\,sf_g(s)\int_0^1 dz\,z\{2n_fP_{qg}(z)+P_{gg}(z)\}
\end{aligned}
$$

なので，

$$
\begin{aligned}
\int_0^1 dz\,z\{2n_fP_{qg}(z)+P_{gg}(z)\} &= \int_0^1 dz\,z\{P_{gq}(z)+P_{qq}(z)\} \\
&= 3\int_0^1 dz\left[1+(1-z)^2+\frac{z+z^3}{(1-z)_+}\right]+2 \\
&= 3\int_0^1 dz\left[z^2-2z+2+\frac{z^3+z-2}{1-z}\right]+2 \\
&= 3\int_0^1 dz\left[z^2-2z+2-(z^2+z+2)\right]+2 \\
&= -4\int_0^1 z\,dz+2 \\
&= 0.
\end{aligned}
$$

Chapter 18

Operator Products and Effective Vertices

18.5 Operator Analysis of Deep Inelastic Scattering

(18.136)

$$\bar{u}(P)\gamma^{\mu}u(P) = \mathrm{Tr}[\bar{u}(P)\gamma^{\mu}u(P)] = \mathrm{Tr}[\gamma^{\mu}u(P)\bar{u}(P)].$$

スピンに関して平均を取って

$$\rightarrow \frac{1}{2}\,\mathrm{Tr}[\gamma^{\mu}\not{P}] = 2P^{\mu}.$$

(18.208)

(18.207) から

$$\frac{d}{d\log Q^2}M_n^{+} = \frac{d}{d\log Q^2}\int_0^1 dx\,x^{n-1}\sum_f (f_f + f_{\bar{f}})(x).$$

(17.128) から

$$= \frac{\alpha_s}{2\pi}\int_0^1 dx\,x^{n-1}\int_x^1 \frac{dz}{z}\left[P_{q\leftarrow q}(z)\sum_f (f_f + f_{\bar{f}})(x/z) + 2P_{q\leftarrow g}(z)f_g(x/z)\right].$$

(17.17)(18.199) から

$$\begin{aligned}
&= \frac{2}{b_0\log(Q^2/\Lambda^2)}\int_0^1 dz\,z^{n-1}P_{q\leftarrow q}(z)\int_0^1 dy\,y^{n-1}\sum_f (f_f + f_{\bar{f}})(y) \\
&\quad + \frac{2}{b_0\log(Q^2/\Lambda^2)}\int_0^1 dz\,z^{n-1}2P_{q\leftarrow g}(z)\int_0^1 dy\,y^{n-1}f_g(y) \\
&= \frac{2}{b_0\log(Q^2/\Lambda^2)}\left[M_n^{+}\int_0^1 dz\,z^{n-1}P_{q\leftarrow q}(z) + M_{gn}\int_0^1 dz\,z^{n-1}2P_{q\leftarrow g}(z)\right].
\end{aligned}$$

(17.129)(18.181)(18.203) から

$$= \frac{2}{b_0\log(Q^2/\Lambda^2)}\left[\frac{a_{ff}^{n}}{4}M_n^{+} + \frac{a_{fg}^{n}}{4}M_{gn}\right].$$

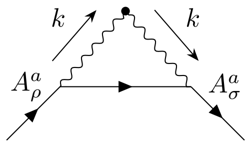
Problems

Problem 18.3: Anomalous dimensions of gluon twsit-2 operators **（計算合わない）**

[5] を参考にして修正予定.

a_{gf}^n の 1 ループ補正

Figure 18.15 (a) のダイアグラムを計算する.



(18.174) から

$$\mathcal{O}_g^{(n)\mu_1\cdots\mu_n} \to -(\partial^{\mu_1}A^\nu - \partial^\nu A^{\mu_1})(i\partial^{\mu_2})\cdots(i\partial^{\mu_{n-1}})(\partial^{\mu_n}A_\nu - \partial_\nu A^{\mu_n}).$$

$\partial^{\mu_1}A^\nu$ と A_σ および $\partial^{\mu_n}A_\nu$ と A_ρ を縮約した場合は,

$$\mathcal{O} = -(ik^{\mu_1})(k^{\mu_2}\cdots k^{\mu_{n-1}})(-ik^{\mu_n})$$

なので,

$$-(ig)^2\int\frac{d^dk}{(2\pi)^d}t^a\gamma^\nu\frac{i}{\not p - \not k}t^a\gamma_\nu\left(\frac{-i}{k^2}\right)^2 k^{\mu_1}\cdots k^{\mu_n}.$$

(6.42)(A.34)(A.37) から $\ell = k - xp$ として

$$\begin{aligned}
&= 3ig^2\int\frac{d^d\ell}{(2\pi)^d}\gamma^\nu\gamma_\alpha\gamma_\nu\int_0^1 dx\,2(1-x)\frac{1}{(\ell^2-\Delta)^3}[\ell-(1-x)p]^\alpha(\ell+xp)^{\mu_1}\cdots(\ell+xp)^{\mu_n}\\
&= \frac{8}{3}(2-d)ig^2\int_0^1 dx\,(1-x)\gamma_\alpha\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}[\ell-(1-x)p]^\alpha(\ell+xp)^{\mu_1}\cdots(\ell+xp)^{\mu_n}.
\end{aligned}$$

このうち (18.161) の $\mathcal{O}_f$ の形 $\gamma^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}$ を持つ項は

$$\begin{aligned}
&\to \frac{8}{3}(2-d)ig^2 n\int_0^1 dx\,(1-x)\gamma_\alpha\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}\ell^\alpha\ell^{\mu_1}x^{n-1}p^{\mu_2}\cdots p^{\mu_n}\\
&= \frac{8}{3}\frac{2-d}{d}ig^2 n\int_0^1 dx\,(1-x)x^{n-1}\gamma_\alpha\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3}g^{\alpha\mu_1}p^{\mu_2}\cdots p^{\mu_n}\\
&= \frac{8}{3}\frac{2-d}{d}ig^2 n\int_0^1 dx\,(1-x)x^{n-1}\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-d/2)}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-d/2}\gamma^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\\
&= 3\frac{g^2}{(4\pi)^2}n\int_0^1 dx\,(1-x)x^{n-1}\gamma^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\frac{2}{\epsilon}\\
&= 3\frac{g^2}{(4\pi)^2}\frac{1}{n+1}\gamma^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\frac{2}{\epsilon}.
\end{aligned}$$

$-\partial^\nu A^{\mu_1}$ と A_σ および $-\partial_\nu A^{\mu_n}$ と A_ρ を縮約した場合は,

$$-(ig)^2\int\frac{d^dk}{(2\pi)^d}t^a\gamma^{\mu_1}\frac{i}{\not p - \not k}t^a\gamma^{\mu_n}\left(\frac{-i}{k^2}\right)^2 k^\nu k^{\mu_2}\cdots k^{\mu_{n-1}}k_\nu$$

$$
\begin{aligned}
&= 3ig^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}[\ell-(1-x)p]^\alpha(\ell+xp)^{\mu_2}\cdots(\ell+xp)^{\mu_{n-1}}\gamma^{\mu_1}\gamma_\alpha\gamma^{\mu_n}\\
&\to 3ig^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}[\ell-(1-x)p]^\alpha(\ell+xp)^{\mu_2}\cdots(\ell+xp)^{\mu_{n-1}}\gamma^{\{\mu_1}\gamma_\alpha\gamma^{\mu_n\}}\\
&\to 3ig^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}[\ell-(1-x)p]^\alpha(\ell+xp)^{\mu_2}\cdots(\ell+xp)^{\mu_{n-1}}(\delta^{\mu_n}_\alpha\gamma^{\mu_1}+\delta^{\mu_1}_\alpha\gamma^{\mu_n})\\
&\to \frac{8}{3}ig^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}\gamma^{\mu_n}[\ell-(1-x)p]^{\mu_1}(\ell+xp)^{\mu_2}\cdots(\ell+xp)^{\mu_{n-1}}\\
&= -\frac{8}{3}ig^2\int_0^1 dx\,(1-x)x^{n-2}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}\gamma^{\mu_n}p^{\mu_1}\cdots p^{\mu_{n-1}}\\
&= \frac{8}{3}\frac{g^2}{(4\pi)^2}\frac{1}{n(n-1)}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n}\frac{2}{\epsilon}.
\end{aligned}
$$

$\partial^{\mu_1}A^\nu$ と A_σ および $-\partial_\nu A^{\mu_n}$ と A_ρ を縮約した場合は，

$$
\begin{aligned}
&(ig)^2\int\frac{d^dk}{(2\pi)^d}t^a\gamma^\nu\frac{i}{\not p-\not k}t^a\gamma^{\mu_n}\left(\frac{-i}{k^2}\right)^2k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu\\
&= -3ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}(k-p)^\alpha k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu\gamma^\nu\gamma_\alpha\gamma^{\mu_n}.
\end{aligned}
$$

$-\partial^\nu A^{\mu_1}$ と A_σ および $\partial^{\mu_n}A_\nu$ と A_ρ を縮約した場合は，

$$
\begin{aligned}
&(ig)^2\int\frac{d^dk}{(2\pi)^d}t^a\gamma^{\mu_1}\frac{i}{\not p-\not k}t^a\gamma_\nu\left(\frac{-i}{k^2}\right)^2k^{\mu_2}\cdots k^{\mu_n}k^\nu\\
&\to -3ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}(k-p)^\alpha k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu\gamma^{\mu_n}\gamma_\alpha\gamma^\nu.
\end{aligned}
$$

以上 2 つを足して，

$$
\begin{aligned}
&-3ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}(k-p)^\alpha k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu(\gamma^{\mu_n}\gamma_\alpha\gamma^\nu+\gamma^\nu\gamma_\alpha\gamma^{\mu_n})\\
&= -\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}(k-p)^\alpha k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu(\delta^\nu_\alpha\gamma^{\mu_n}+\delta^{\mu_n}_\alpha\gamma^\nu-g^{\mu_n\nu}\gamma_\alpha)\\
&\to -\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}\gamma^{\mu_n}k\cdot(k-p)\\
&\quad -\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}(k^{\mu_n}-p^{\mu_n})\not k\\
&\quad +\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_n}(\not k-\not p)\\
&= -\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}\gamma^{\mu_n}k\cdot(k-p)\\
&\quad +\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}p^{\mu_n}\not k\\
&= -\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}\gamma^{\mu_n}\\
&= +\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}\gamma^{\mu_n}k\cdot p
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8}{3}ig^2\int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{k^2}\right)^2\frac{1}{(k-p)^2}k^{\mu_1}\cdots k^{\mu_{n-1}}p^{\mu_n}\not{k} \\
\to &-\frac{8}{3}ig^2\int dx\,x^{n-1}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n} \\
&+\frac{16}{3}\frac{n-1}{d}ig^2\int dx\,(1-x)x^{n-2}\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n} \\
&+\frac{16}{3}\frac{n-1}{d}ig^2\int dx\,(1-x)x^{n-2}\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n} \\
=&\,\frac{8}{3}\frac{g^2}{(4\pi)^2}\int dx\,x^{n-1}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n}\frac{2}{\epsilon} \\
&-\frac{8}{3}(n-1)\frac{g^2}{(4\pi)^2}\int dx\,(1-x)x^{n-2}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n}\frac{2}{\epsilon} \\
=&\,0.
\end{aligned}
$$

縮約を逆にしたものも含めて，1 ループの補正は

$$
\begin{aligned}
&\frac{8}{3}\frac{g^2}{(4\pi)^2}\frac{1}{n+1}\gamma^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\frac{2}{\epsilon}+\frac{16}{3}\frac{g^2}{(4\pi)^2}\frac{1}{n(n-1)}p^{\mu_1}\cdots p^{\mu_{n-1}}\gamma^{\mu_n}\frac{2}{\epsilon} \\
&=\frac{8}{3}\frac{g^2}{(4\pi)^2}\frac{n^2+n+2}{n(n^2-1)}\gamma^{\mu_1}p^{\mu_2}\cdots p^{\mu_n}\frac{2}{\epsilon} \\
&\to\frac{8}{3}\frac{g^2}{(4\pi)^2}\frac{n^2+n+2}{n(n^2-1)}\mathcal{O}_f^{(n)\mu_1\cdots\mu_n}\left(\frac{2}{\epsilon}-\log M^2+\cdots\right).
\end{aligned}
$$

a_{gg}^n の 1 ループ補正

$\mathcal{O}_g$ のゲージ場への作用

$$
\begin{aligned}
\langle\Omega|\,A_b^\sigma\mathcal{O}_g^{(n)}A_a^\rho\,|\Omega\rangle &= \langle 0|\,A_b^\sigma\exp\left(i\int d^4x\,\mathcal{L}\right)\mathcal{O}_g^{(n)}A_a^\rho\,|0\rangle \\
&= \langle 0|\,A_b^\sigma\mathcal{O}_g^{(n)}A_a^\rho\,|0\rangle+\langle 0|\,A_b^\sigma\left(i\int d^4x\,\mathcal{L}\right)\mathcal{O}_g^{(n)}A_a^\rho\,|0\rangle+\cdots
\end{aligned}
$$

を g^2 のオーダーで考える．

(18.167) の後の議論と同様に，随伴表現を使えば

$$(iD^{\mu_j})_{cd}=i\partial_{\mu_j}\delta_{cd}-gA_e^{\mu_j}(t_G^e)_{cd}=i\partial_{\mu_j}\delta_{cd}+igA_e^{\mu_j}f^{cde}$$

である．さらに (16.2) から

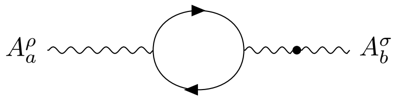
$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu.$$

(18.174) は

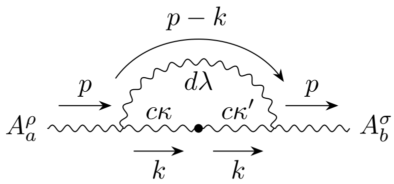
$$
\begin{aligned}
\mathcal{O}_g^{(n)\mu_1\cdots\mu_n} &= -(\partial^{\mu_1}A_c^\nu-\partial^\nu A_c^{\mu_1})(i\partial^{\mu_2})\cdots(i\partial^{\mu_{n-1}})(\partial^{\mu_n}A_\nu^c-\partial_\nu A_c^{\mu_n}) \\
&-\sum_{j=2}^{n-1}(\partial^{\mu_1}A_c^\nu-\partial^\nu A_c^{\mu_1})(i\partial^{\mu_2})\cdots(igA_e^{\mu_j}f^{cde})\cdots(i\partial^{\mu_{n-1}})(\partial^{\mu_n}A_\nu^d-\partial_\nu A_d^{\mu_n}) \\
&-(\partial^{\mu_1}A_c^\nu-\partial^\nu A_c^{\mu_1})(i\partial^{\mu_2})\cdots(i\partial^{\mu_{n-1}})gf^{cde}A_d^{\mu_n}A_\nu^e \\
&-\cdots
\end{aligned}
$$

となる．第 1 項は 2 つのゲージ場と縮約でき，第 2 項は 3 つのゲージ場と縮約できる．

$\mathcal{O}_g$ の第 1 項は g^0 オーダーなので，$e^{i\mathcal{L}}$ の展開のうち g^2 のオーダーのものを考えれば良い．これは fermion 頂点 2 つ，3-boson 頂点 2 つ，もしくは 4-boson 頂点 1 つである．fermion 頂点 2 つの場合は



となるが，これは external leg correction δ_3 なので，$\delta_{\mathcal{O}}$ には含めない．4-boson 頂点は $\mathcal{O}_g$ の形を持たないので，計算には含めない．結局，$\delta_{\mathcal{O}}$ の計算に含めるのは 3-boson 頂点 2 つからなる Figure 18.15 (b) 1 つ目の diagram である．



頂点のゲージ場 A^κ_c, $A^{\kappa'}_c$ と

$$\mathcal{O}_g^{(n)\mu_1\cdots\mu_n} \to (i\partial^{\mu_1}A^\nu_c - i\partial^\nu A^{\mu_1}_c)(i\partial^{\mu_2})\cdots(i\partial^{\mu_{n-1}})(i\partial^{\mu_n}A^c_\nu - i\partial_\nu A^{\mu_n}_c)$$

のゲージ場との縮約を計算する．

$\partial^{\mu_1}\overbracket{A^\nu_c A^{\kappa'}_c}$, $\partial^{\mu_n}\overbracket{A^c_\nu A^\kappa_c}$ の項は

$$\begin{aligned}
&g^2 f^{adc}f^{bcd}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)^2\frac{-i}{(k-p)^2}\\
&\quad\times[2k^2g^{\rho\sigma} - 2(k\cdot p)g^{\rho\sigma} + 5p^2g^{\rho\sigma} + 10k^\rho k^\sigma - 5k^\rho p^\sigma - 5k^\sigma p^\rho - 2p^\rho p^\sigma]k^{\mu_1}\cdots k^{\mu_n}\\
&\to -3ig^2\delta^{ab}\int_0^1 dx\,2(1-x)\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3}\left[2x^n - \frac{2n}{d}x^{n-1} + \frac{10}{d}x^n\right]g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\\
&= -3ig^2\delta^{ab}\int_0^1 dx\,(1-x)(9x^n - nx^{n-1})\frac{i}{(4\pi)^2}g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}\\
&= 3\delta^{ab}\frac{g^2}{(4\pi)^2}\left(-\frac{9}{n+2} + \frac{8}{n+1}\right)g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}.
\end{aligned}$$

$-\partial^\nu\overbracket{A^{\mu_1}_c A^{\kappa'}_c}$, $-\partial_\nu\overbracket{A^{\mu_n}_c A^\kappa_c}$ の項は

$$\begin{aligned}
&g^2 f^{adc}f^{bcd}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)^2\frac{-i}{(k-p)^2}N^{\rho\sigma\mu_1\mu_n}k^\nu k^{\mu_2}\cdots k^{\mu_{n-1}}k_\nu\\
&= -3ig^2\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p)^2}N^{\rho\sigma\mu_1\mu_n}k^{\mu_2}\cdots k^{\mu_{n-1}}.
\end{aligned}$$

$N^{\rho\sigma\mu_1\mu_n}$ のうち $g^{\rho\sigma}p^{\mu_1}p^{\mu_n}$ の形の項は

$$\begin{aligned}
N^{\rho\sigma\mu_1\mu_n} &= k^2g^{\rho\mu_1}g^{\sigma\mu_n} + 2(k\cdot p)g^{\rho\mu_1}g^{\sigma\mu_n} + p^2g^{\rho\mu_1}g^{\sigma\mu_n}\\
&\quad + g^{\rho\sigma}k^{\mu_n}k^{\mu_1} - 2g^{\rho\sigma}k^{\mu_n}p^{\mu_1} - 2g^{\rho\sigma}k^{\mu_1}p^{\mu_n} + 4g^{\rho\sigma}p^{\mu_n}p^{\mu_1}\\
&\quad - 2g^{\rho\mu_n}k^\sigma k^{\mu_1} + 4g^{\rho\mu_n}k^\sigma p^{\mu_1} + g^{\rho\mu_n}k^{\mu_1}p^\sigma - 2g^{\rho\mu_n}p^\sigma p^{\mu_1}\\
&\quad - g^{\rho\mu_1}k^\sigma k^{\mu_n} - 4g^{\rho\mu_1}k^\sigma p^{\mu_n} + 2g^{\rho\mu_1}k^{\mu_n}p^\sigma - g^{\rho\mu_1}p^\sigma p^{\mu_n}
\end{aligned}$$

$$
\begin{aligned}
&- g^{\sigma\mu_n}k^\rho k^{\mu_1} - 4g^{\sigma\mu_n}k^\rho p^{\mu_1} + 2g^{\sigma\mu_n}k^{\mu_1}p^\rho - g^{\sigma\mu_n}p^\rho p^{\mu_1} \\
&- 2g^{\sigma\mu_1}k^\rho k^{\mu_n} + 4g^{\sigma\mu_1}k^\rho p^{\mu_n} + g^{\sigma\mu_1}k^{\mu_n}p^\rho - 2g^{\sigma\mu_1}p^\rho p^{\mu_n} \\
&+ 4g^{\mu_1\mu_n}k^\rho k^\sigma - 2g^{\mu_1\mu_n}k^\rho p^\sigma - 2g^{\mu_1\mu_n}k^\sigma p^\rho + g^{\mu_1\mu_n}p^\rho p^\sigma \\
\to\ & g^{\rho\sigma}k^{\mu_n}k^{\mu_1} - 2g^{\rho\sigma}k^{\mu_n}p^{\mu_1} - 2g^{\rho\sigma}k^{\mu_1}p^{\mu_n} + 4g^{\rho\sigma}p^{\mu_n}p^{\mu_1} \\
\to\ & (x^2 - 4x + 4)g^{\rho\sigma}p^{\mu_1}p^{\mu_n}
\end{aligned}
$$

なので,

$$
\begin{aligned}
&= -3ig^2\delta^{ab}\int_0^1 dx\, x^{n-2}(x^2 - 4x + 4)\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2 - \Delta)^2}g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n} \\
&= 3\delta^{ab}\frac{g^2}{(4\pi)^2}\left(\frac{1}{n+1} - \frac{4}{n} + \frac{4}{n-1}\right)g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}.
\end{aligned}
$$

$\partial^{\mu_1}\overbracket{A_c^{\nu}A}{}_c^{\kappa'}$, $-\partial_\nu \overbracket{A_c^{\mu_n}A}{}_c^{\kappa}$ の項は

$$
\begin{aligned}
&- g^2 f^{adc}f^{bcd}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)^2\frac{-i}{(k-p)^2}N^{\rho\sigma\nu\mu_n}k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu \\
&= 3ig^2\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^4(k-p)^2}N^{\rho\sigma\nu\mu_n}k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu.
\end{aligned}
$$

$-\partial^\nu \overbracket{A_c^{\mu_1}A}{}_c^{\kappa'}$, $\partial^{\mu_n}\overbracket{A_\nu^{c}A}{}_c^{\kappa}$ の項は

$$
\begin{aligned}
&- g^2 f^{adc}f^{bcd}\int\frac{d^dk}{(2\pi)^d}\left(\frac{-i}{k^2}\right)^2\frac{-i}{(k-p)^2}N^{\rho\sigma\mu_1}{}_\nu k^{\mu_2}\cdots k^{\mu_n}k^\nu \\
&\to 3ig^2\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^4(k-p)^2}N^{\rho\sigma\mu_n\nu}k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu.
\end{aligned}
$$

$N^{\rho\sigma(\nu\mu_n)}$ のうち $g^{\rho\sigma}p^{\mu_1}p^{\mu_n}$ の形の項は

$$
\begin{aligned}
&N^{\rho\sigma\nu\mu_n} + N^{\rho\sigma\mu_n\nu} \\
&= k^2g^{\rho\mu_n}g^{\sigma\nu} + k^2g^{\rho\nu}g^{\sigma\mu_n} + 2(k\cdot p)g^{\rho\mu_n}g^{\sigma\nu} + 2(k\cdot p)g^{\rho\nu}g^{\sigma\mu_n} + p^2g^{\rho\mu_n}g^{\sigma\nu} + p^2g^{\rho\nu}g^{\sigma\mu_n} \\
&\quad + 2g^{\rho\sigma}k^{\mu_n}k^\nu - 4g^{\rho\sigma}k^{\mu_n}p^\nu - 4g^{\rho\sigma}k^\nu p^{\mu_n} + 8g^{\rho\sigma}p^{\mu_n}p^\nu \\
&\quad - 3g^{\rho\mu_n}k^\sigma k^\nu + 3g^{\rho\mu_n}k^\nu p^\sigma - 3g^{\rho\mu_n}p^\sigma p^\nu \\
&\quad - 3g^{\rho\nu}k^\sigma k^{\mu_n} + 3g^{\rho\nu}k^{\mu_n}p^\sigma - 3g^{\rho\nu}p^\sigma p^{\mu_n} \\
&\quad - 3g^{\sigma\mu_n}k^\rho k^\nu + 3g^{\sigma\mu_n}k^\nu p^\rho - 3g^{\sigma\mu_n}p^\rho p^\nu \\
&\quad - 3g^{\sigma\nu}k^\rho k^{\mu_n} + 3g^{\sigma\nu}k^{\mu_n}p^\rho - 3g^{\sigma\nu}p^\rho p^{\mu_n} \\
&\quad + 8g^{\mu_n\nu}k^\rho k^\sigma - 4g^{\mu_n\nu}k^\rho p^\sigma - 4g^{\mu_n\nu}k^\sigma p^\rho + 2g^{\mu_n\nu}p^\rho p^\sigma \\
&\to 2g^{\rho\sigma}k^{\mu_n}k^\nu - 4g^{\rho\sigma}k^{\mu_n}p^\nu - 4g^{\rho\sigma}k^\nu p^{\mu_n} + 8g^{\rho\sigma}p^{\mu_n}p^\nu - 3g^{\rho\nu}k^\sigma k^{\mu_n} - 3g^{\sigma\nu}k^\rho k^{\mu_n} + 8g^{\mu_n\nu}k^\rho k^\sigma.
\end{aligned}
$$

従って,

$$
\begin{aligned}
&(N^{\rho\sigma\nu\mu_n} + N^{\rho\sigma\mu_n\nu})k^{\mu_1}\cdots k^{\mu_{n-1}}k_\nu \\
&\to 2k^2g^{\rho\sigma}k^{\mu_1}\cdots k^{\mu_n} - 4(k\cdot p)g^{\rho\sigma}k^{\mu_1}\cdots k^{\mu_n} \\
&\quad - 4k^2g^{\rho\sigma}k^{\mu_1}\cdots k^{\mu_{n-1}}p^{\mu_n} + 8(k\cdot p)g^{\rho\sigma}k^{\mu_1}\cdots k^{\mu_{n-1}}p^{\mu_n} \\
&\quad + 2k^\rho k^\sigma k^{\mu_1}\cdots k^{\mu_n} \\
&= \left[2x^n - \frac{4n}{d}x^{n-1} - 4x^{n-1} + \frac{8(n-1)}{d}x^{n-2} + \frac{2}{d}x^n\right]\ell^2 g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}
\end{aligned}
$$

$$= \left[\frac{5}{2}x^n - (n+4)x^{n-1} + 2(n-1)x^{n-2}\right]\ell^2 g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}.$$

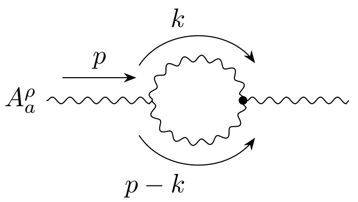
以上から

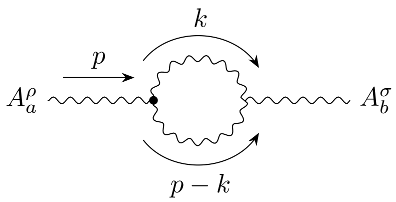
$$\begin{aligned}
&= 3ig^2\delta^{ab}\int_0^1 dx\,(1-x)[5x^n - 2(n+4)x^{n-1} + 4(n-1)x^{n-2}]\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^3}g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n} \\
&= -3\delta^{ab}\frac{g^2}{(4\pi)^2}\left(-\frac{5}{n+2}+\frac{11}{n+1}-\frac{4}{n}\right)g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}.
\end{aligned}$$

これらを全て足せば，（縮約を逆にしたものも含めて 2 倍，Taylor 展開の係数で 1/2 倍）

$$3\delta^{ab}\frac{g^2}{(4\pi)^2}\left[-\frac{4}{n+2}-\frac{2}{n+1}+\frac{4}{n-1}\right]g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}. \qquad [18.5.1]$$

$\mathcal{O}_g$ の第 2 項は g^1 オーダーなので，$e^{i\mathcal{L}}$ の展開のうち g^1 のオーダーのものを考えれば良い．これは 3-boson 頂点 1 つであり，次の diagram で表現される．

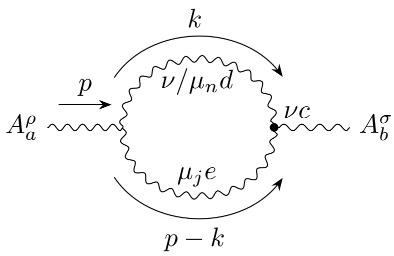




$\mathcal{O}_g$ の形から，2 つの diagram は同じ振幅を与えるので，左側のみ考える．

$$\mathcal{O}_g \to (i\partial^{\mu_1}A_c^\nu - i\partial^\nu A_c^{\mu_1})(i\partial^{\mu_2})\cdots(igA_e^{\mu_j}f^{cde})\cdots(i\partial^{\mu_{n-1}})(i\partial^{\mu_n}A_\nu^d - i\partial_\nu A_d^{\mu_n})$$

のゲージ場のうち，A_b^σ と縮約した際に，$g^{\rho\sigma}$ を与えるのは $\partial^{\mu_1}A_c^\nu$（もしくは $\partial^{\mu_n}A_\nu^d$）である．



$\partial^{\mu_n}A_\nu^d$ を縮約した項は

$$\begin{aligned}
&igf^{cde}\delta^{bc}\int\frac{d^dk}{(2\pi)^d}gf^{ade}[\delta_\nu^\rho(p+k)^{\mu_j} + \delta_\nu^{\mu_j}(p-2k)^\rho + g^{\mu_j\rho}(k-2p)_\nu]g^{\nu\sigma} \\
&\quad\times\frac{-i}{k^2}\frac{-i}{(p-k)^2}p^{\mu_1}\cdots p^{\mu_{j-1}}k^{\mu_{j+1}}\cdots k^{\mu_n} \\
&\to -3ig^2g^{\rho\sigma}\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k-p)^2}p^{\mu_1}\cdots p^{\mu_{j-1}}(k+p)^{\mu_j}k^{\mu_{j+1}}\cdots k^{\mu_n} \\
&= -3ig^2g^{\rho\sigma}\delta^{ab}\int_0^1 dx\,(1+x)x^{n-j}\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^2}p^{\mu_1}\cdots p^{\mu_n} \\
&= 3\delta^{ab}\frac{g^2}{(4\pi)^2}\left(\frac{1}{n-j+1}+\frac{1}{n-j+2}\right)g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}.
\end{aligned}$$

$-\partial_\nu A_d^{\mu_n}$ を縮約した項は 0.

$F^{\mu_1\nu}$, $F^{\mu_n}{}_\nu$ からゲージ場を 2 つ取り出した項はそれぞれ $j=1,n$ を与える．さらに diagram の対称性は $S=2$ なので,

$$\begin{aligned}
&-3\delta^{ab}\frac{g^2}{(4\pi)^2}2\cdot 2\cdot\frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{n-j+1}+\frac{1}{n-j+2}\right)g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}\\
&=-3\delta^{ab}\frac{g^2}{(4\pi)^2}\left(4\sum_{j=2}^{n}\frac{1}{j}+\frac{2}{n+1}+2\right)g^{\rho\sigma}p^{\mu_1}\cdots p^{\mu_n}\frac{2}{\epsilon}.
\end{aligned} \tag{[18.5.2]}$$

[18.5.1][18.5.2] を足して,

$$-\delta_{\mathcal{O}}=3\frac{g^2}{(4\pi)^2}\left[-2-4\sum_{j=2}^{n}\frac{1}{j}-\frac{4}{n+2}-\frac{4}{n+1}+\frac{4}{n-1}\right]\left(\frac{2}{\epsilon}-\log M^2\right).$$

(18.23)(16.74) から

$$\begin{aligned}
\gamma_{\mathcal{O}}&=M\frac{\partial}{\partial M}(-\delta_{\mathcal{O}}+\delta_3)\\
&=\frac{6g^2}{(4\pi)^2}\left(\frac{1}{3}+\frac{2}{9}n_f+4\sum_{j=2}^{n}\frac{1}{j}+\frac{4}{n+2}+\frac{4}{n+1}-\frac{4}{n-1}\right).
\end{aligned}$$

Chapter 19

Perturbation Theory Anomalies

19.1 The Axial Current in Two Dimensions

(19.15)

ψ は反交換することに注意して，(4.31) から

$$
\begin{aligned}
\langle j^\mu(q)\rangle &= \int d^2x\, e^{iq\cdot x}\, \langle j^\mu(x)\rangle \\
&= \int d^2x\, e^{iq\cdot x}\, \langle\Omega|\, \bar\psi(x)\gamma^\mu\psi(x)\, |\Omega\rangle \\
&\sim i\int d^2x\, e^{iq\cdot x} \int d^2y\, \langle 0|\, \bar\psi(x)\gamma^\mu\psi(x)\mathcal{L}(y)\, |0\rangle \\
&= -ie\int d^2x \int d^2y\, e^{iq\cdot x}\, \langle 0|\, \bar\psi(x)\gamma^\mu\psi(x)A_\nu(y)\bar\psi(y)\gamma^\nu\psi(y)\, |0\rangle \\
&= -ie\int d^2x \int d^2y\, e^{iq\cdot x}\, \langle 0|\, \bar\psi_\alpha(x)(\gamma^\mu)_{\alpha\beta}\psi_\beta(x)A_\nu(y)\bar\psi_\gamma(y)(\gamma^\nu)_{\gamma\delta}\psi_\delta(y)\, |0\rangle \\
&= ie\int d^2x \int d^2y\, e^{iq\cdot x} \int \frac{d^2k}{(2\pi)^2}\left(\frac{1}{\not{k}}\right)_{\delta\alpha} e^{-ik\cdot(y-x)} \int \frac{d^2k'}{(2\pi)^2}\left(\frac{1}{\not{k}'}\right)_{\beta\gamma} e^{-ik'\cdot(x-y)} \\
&\qquad \times (\gamma^\mu)^{\alpha\beta}(\gamma^\nu)^{\gamma\delta}\int \frac{d^2p}{(2\pi)^2} A_\nu(p)e^{-p\cdot y} \\
&= ie\int \frac{d^2k}{(2\pi)^2}\, \mathrm{Tr}\left[\frac{1}{\not{k}}\gamma^\mu \frac{1}{\not{k}+\not{q}}\gamma^\nu\right] A_\nu(q).
\end{aligned}
$$

これを (7.71) と比べれば良い．

19.3 Goldstone Bosons and Chiral Symmetries in QCD

(19.84)

Lagrangian は

$$
\mathcal{L} = u_L^\dagger i\bar\sigma^\mu D_\mu u_L + d_L^\dagger i\bar\sigma^\mu D_\mu d_L + (R).
$$

$U(1)$ 微小変換は

$$
\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to e^{-i\epsilon}\begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} e^{-i\epsilon}u_L \\ e^{-i\epsilon}d_L \end{pmatrix} \approx \begin{pmatrix} (1-i\epsilon)u_L \\ (1-i\epsilon)d_L \end{pmatrix}
$$

で与えられる．この変換で $\mathcal{L}$ は不変なので，付随するカレントは

$$\epsilon j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu u_L}\Delta u_L + \frac{\partial\mathcal{L}}{\partial\partial_\mu d_L}\Delta d_L = u_L^\dagger i\bar{\sigma}^\mu(-i\epsilon u_L) + d_L^\dagger i\bar{\sigma}^\mu(-i\epsilon d_L).$$

一方で

$$Q_L = \left(\frac{1-\gamma^5}{2}\right)\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma^5}{2}u \\ \frac{1-\gamma^5}{2}d \end{pmatrix}$$

に対し

$$\begin{aligned}
\bar{Q}_L\gamma^\mu Q_L &= \begin{pmatrix} \bar{u}\frac{1+\gamma^5}{2} & \bar{d}\frac{1+\gamma^5}{2} \end{pmatrix}\begin{pmatrix} \gamma^\mu\frac{1-\gamma^5}{2}u \\ \gamma^\mu\frac{1-\gamma^5}{2}d \end{pmatrix} \\
&= \bar{u}\frac{1+\gamma^5}{2}\gamma^\mu\frac{1-\gamma^5}{2}u + (d) \\
&= \begin{pmatrix} u_R^\dagger & u_L^\dagger \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} + (d) \\
&= u_L^\dagger\bar{\sigma}^\mu u_L + (d)
\end{aligned}$$

なので $j^\mu = \bar{Q}_L\gamma^\mu Q_L$.

$SU(2)$ 微小変換は

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \to e^{-i\epsilon_a\tau^a}\begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} e^{-i\epsilon_a\tau^a}u_L \\ e^{-i\epsilon_a\tau^a}d_L \end{pmatrix} \approx \begin{pmatrix} (1-i\epsilon_a\tau^a)u_L \\ (1-i\epsilon_a\tau^a)d_L \end{pmatrix}$$

で与えられる．後は同様.

Chapter 20

Gauge Theories with Spontaneous Symmetry Breaking

20.1 The Higgs Mechanism

(20.27)

$\mathbb{C}^2$ scalar を $\mathbb{R}^4$ vector として表す．(21.38) から

$$\mathbb{C}^2 \ni \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} \mapsto \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} \in \mathbb{R}^4 \tag{20.1.1}$$

とする．この対応により T^a は 4×4 行列となる．(20.14) から

$$T^a = -it^a = -\frac{i}{2}\sigma^a.$$

$a = 1$ なら

$$T^1 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} = -\frac{i}{2}\sigma^1 \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} & -i \\ -i & \end{pmatrix}\begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix}$$
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} \phi^3 - ih \\ -\phi^1 + i\phi^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} h \\ -\phi^3 \\ \phi^2 \\ -\phi^1 \end{pmatrix}$$

なので

$$T^1 = \frac{1}{2}\begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

$a = 2$ なら

$$T^2 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} = -\frac{i}{2}\sigma^2 \frac{1}{\sqrt{2}}\begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix}$$

(20.27)

$$
= \frac{1}{2\sqrt{2}} \begin{pmatrix} -h - i\phi^3 \\ -\phi^2 - i\phi^1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi^3 \\ h \\ -\phi^1 \\ -\phi^2 \end{pmatrix}
$$

なので

$$
T^2 = \frac{1}{2} \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.
$$

$a = 3$ なら

$$
\begin{aligned}
T^3 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} &= -\frac{i}{2}\sigma^3 \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -i & \\ & i \end{pmatrix} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} \\
&= \frac{1}{2\sqrt{2}} \begin{pmatrix} -\phi^1 + i\phi^2 \\ -\phi^3 + ih \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\phi^2 \\ \phi^1 \\ h \\ -\phi^3 \end{pmatrix}
\end{aligned}
$$

なので

$$
T^3 = \frac{1}{2} \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}.
$$

以上から

$$
T^1 = \frac{1}{2} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}. \qquad [20.1.2]
$$

これらを

$$
\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ 0 \end{pmatrix} \in \mathbb{R}^3
$$

に制限すれば

$$
T^1 = \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix}, \quad T^2 = \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \quad T^3 = \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}. \qquad [20.1.3]
$$

特に，$a = 1, 2, 3$ に対して

$$
(T^a)_{bc} = \epsilon^{bac}.
$$

(20.27) 右辺は $\phi^c \in \mathbb{R}^3$ に変換した vector．$\phi \in \mathbb{C}^2$ は $SU(2)$ の基本表現に属するが，$\mathbb{R}^3$ vector への変換によって（見かけ上）随伴表現に属している．

20.2 The Glashow-Weinberg-Salam Theory of Weak Interactions

(20.80)

(20.63)(20.64) から

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2), \quad Z_\mu^0 = \frac{1}{\sqrt{g^2+g'^2}}(gA_\mu^3 - g'B_\mu), \quad A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(g'A_\mu^3 + gB_\mu).$$

(20.68)(20.70) から

$$e = \frac{gg'}{\sqrt{g^2+g'^2}}, \quad \cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}, \quad \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}.$$

さらに

$$A_\mu^3 = A_\mu \sin\theta_w + Z_\mu^0 \cos\theta_w, \quad B_\mu = A_\mu\cos\theta_w - Z_\mu^0 \sin\theta_w,$$

E_L の項は

$$\begin{aligned}
&\begin{pmatrix}\bar\nu_L & \bar e_L\end{pmatrix} i\gamma^\mu \left(-igA_\mu^a\tau^a + i\frac{1}{2}g'B_\mu\right)\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} \\
&= \begin{pmatrix}\bar\nu_L & \bar e_L\end{pmatrix} i\gamma^\mu \left[-i\frac{g}{\sqrt{2}}\begin{pmatrix} & W_\mu^+ \\ W_\mu^- & \end{pmatrix} - i\frac{g}{2}A_\mu^3\begin{pmatrix}1 & \\ & -1\end{pmatrix} + i\frac{g'}{2}B_\mu\begin{pmatrix}1 & \\ & 1\end{pmatrix}\right]\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} \\
&= \frac{g}{\sqrt{2}}W_\mu^+(\bar\nu_L\gamma^\mu e_L) + \frac{g}{\sqrt{2}}W_\mu^-(\bar e_L\gamma^\mu\nu_L) \\
&\quad + \frac{1}{2}\bar\nu_L\gamma^\mu(gA_\mu^3 - g'B_\mu)\nu_L - \frac{1}{2}\bar e_L\gamma^\mu(gA_\mu^3 + g'B_\mu)e_L.
\end{aligned}$$

第 1, 2 項は $J_W^{\mu+}$, $J_W^{\mu-}$ の第 1 項を与える．第 3 項は

$$\frac{1}{2}\bar\nu_L\gamma^\mu(gA_\mu^3 - g'B_\mu)\nu_L = \frac{\sqrt{g^2+g'^2}}{2}Z_\mu^0\bar\nu_L\gamma^\mu\nu_L = \frac{g}{2}\frac{Z_\mu^0}{\cos\theta_w}(\bar\nu_L\gamma^\mu\nu_L)$$

なので J_Z^μ の第 1 項を与える．第 4 項は

$$\begin{aligned}
&-\frac{1}{2}\bar e_L\gamma^\mu(gA_\mu^3 + g'B_\mu)e_L \\
&= -\frac{\sqrt{g^2+g'^2}}{2}(A_\mu^3\cos\theta_w + B_\mu\sin\theta_w)(\bar e_L\gamma^\mu e_L) \\
&= -\frac{\sqrt{g^2+g'^2}}{2}\left[2A_\mu\frac{e}{\sqrt{g^2+g'^2}} + Z_\mu^0(1-2\sin^2\theta_w)\right](\bar e_L\gamma^\mu e_L) \\
&= -eA_\mu(\bar e_L\gamma^\mu e_L) + g\frac{Z_\mu^0}{\cos\theta_w}\left(-\frac{1}{2} + \sin^2\theta\right)(\bar e_L\gamma^\mu e_L)
\end{aligned}$$

なので J_{EM}^μ の第 1 項の左成分と J_Z^μ の第 2 成分を与える．

e_R の項は

$$\begin{aligned}
\bar e_R i\gamma^\mu(ig'B_\mu)e_R &= g'(Z_\mu^0\sin\theta_w - A_\mu\cos\theta_w)(\bar e_R\gamma^\mu e_R) \\
&= g\frac{\sin^2\theta_w}{\cos\theta_w}Z_\mu^0(\bar e_R\gamma^\mu e_R) - eA_\mu(\bar e_R\gamma^\mu e_R)
\end{aligned}$$

なので J_Z^μ の第 3 項と J_{EM}^μ の第 1 項の右成分を与える．u_R は $Y=2/3$，d_R は $Y=-1/3$ とすれば良い．

Q_L の項は

$$
\begin{aligned}
&\begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix} i\gamma^\mu \left(-igA_\mu^a\tau^a - i\frac{1}{6}g'B_\mu\right)\begin{pmatrix}u_L\\ d_L\end{pmatrix}\\
&=\begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix} i\gamma^\mu \left[-i\frac{g}{\sqrt{2}}\begin{pmatrix} & W_\mu^+\\ W_\mu^- & \end{pmatrix} - i\frac{g}{2}A_\mu^3\begin{pmatrix}1 & \\ & -1\end{pmatrix} - i\frac{g'}{6}B_\mu\begin{pmatrix}1 & \\ & 1\end{pmatrix}\right]\begin{pmatrix}u_L\\ d_L\end{pmatrix}\\
&=\frac{g}{\sqrt{2}}W_\mu^+(\bar{u}_L\gamma^\mu d_L) + \frac{g}{\sqrt{2}}W_\mu^-(\bar{d}_L\gamma^\mu u_L)\\
&\quad+\frac{1}{2}\bar{u}_L\gamma^\mu\left(gA_\mu^3 + \frac{1}{3}g'B_\mu\right)u_L - \frac{1}{2}\bar{d}_L\gamma^\mu\left(gA_\mu^3 - \frac{1}{3}g'B_\mu\right)d_L.
\end{aligned}
$$

第 1, 2 項は $J_W^{\mu+}$, $J_W^{\mu-}$ の第 2 項を与える．第 3 項は

$$
\begin{aligned}
&\frac{1}{2}\bar{u}_L\gamma^\mu\left(gA_\mu^3 + \frac{1}{3}g'B_\mu\right)u_L\\
&=\frac{\sqrt{g^2+g'^2}}{2}\left(A_\mu^3\cos\theta_w + \frac{1}{3}B_\mu\sin\theta_w\right)(\bar{u}_L\gamma^\mu u_L)\\
&=\frac{g}{2\cos\theta_w}\left[\frac{4}{3}\cos\theta_w\sin\theta_w A_\mu + Z_\mu^0\left(\cos^2\theta_w - \frac{\sin^2\theta_w}{3}\right)\right](\bar{u}_L\gamma^\mu u_L)\\
&=\frac{2}{3}eA_\mu(\bar{u}_L\gamma^\mu u_L) + \frac{g}{2}\frac{Z_\mu^0}{\cos\theta_w}\left(1-\frac{4}{3}\sin^2\theta_w\right)(\bar{u}_L\gamma^\mu u_L)
\end{aligned}
$$

なので J_{EM}^μ の第 2 項と J_Z^μ の第 4 項を与える．第 4 項は

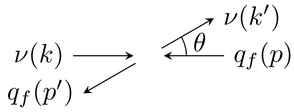
$$
\begin{aligned}
&-\frac{1}{2}\bar{d}_L\gamma^\mu\left(gA_\mu^3 - \frac{1}{3}g'B_\mu\right)d_L\\
&=\frac{\sqrt{g^2+g'^2}}{2}\left(-A_\mu^3\cos\theta_w + \frac{1}{3}B_\mu\sin\theta_w\right)(\bar{d}_L\gamma^\mu d_L)\\
&=\frac{g}{2\cos\theta_w}\left[-\frac{2}{3}\cos\theta_w\sin\theta_w A_\mu - Z_\mu^0\left(\cos^2\theta_w + \frac{\sin^2\theta_w}{3}\right)\right](\bar{d}_L\gamma^\mu d_L)\\
&=-\frac{1}{3}eA_\mu(\bar{u}_L\gamma^\mu u_L) + \frac{g}{2}\frac{Z_\mu^0}{\cos\theta_w}\left(-1+\frac{2}{3}\sin^2\theta_w\right)(\bar{u}_L\gamma^\mu u_L)
\end{aligned}
$$

なので J_{EM}^μ の第 3 項と J_Z^μ の第 5 項を与える．

Problems

Problem 20.4: Neutral-current deep inelastic scattering

parton レベルの散乱は次のようになる．



Mandelstam variable は

$$
p\cdot k = p'\cdot k' = \frac{\hat{s}}{2},\quad p\cdot p' = k\cdot k' = -\frac{\hat{t}}{2},\quad p\cdot k' = p'\cdot k = -\frac{\hat{u}}{2}.
$$

(17.31)(20.80) から effective Lagrangian は

$$\Delta\mathcal{L} = \frac{g^2}{\cos^2\theta_w}\frac{1}{m_Z{}^2}\left[\bar{\nu}\gamma_\mu\frac{1}{2}\frac{1-\gamma^5}{2}\nu\right]\left[\bar{q}\gamma^\mu(T^3-\sin^2\theta_w Q)\frac{1\mp\gamma^5}{2}q\right].$$

$\nu + u_L \to \nu + u_L$

不変振幅は

$$i\mathcal{M} = \frac{g^2}{m_Z{}^2\cos^2\theta_w}\frac{\frac{1}{2}-\frac{2}{3}\sin^2\theta_w}{2}\left[\bar{\nu}(k')\gamma_\mu\frac{1-\gamma^5}{2}\nu(k)\right]\left[\bar{u}(p')\gamma^\mu\frac{1-\gamma^5}{2}u(p)\right]$$

なので (A.27) から

$$\begin{aligned}
|\mathcal{M}|^2 &= \sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{4}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2 \mathrm{Tr}\left[\not{k}'\gamma_\mu\frac{1-\gamma^5}{2}\not{k}\gamma_\nu\right]\mathrm{Tr}\left[\not{p}'\gamma^\mu\frac{1-\gamma^5}{2}\not{p}\gamma^\nu\right] \\
&= \left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2\left[k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu} + ik'_\alpha k_\beta\epsilon^{\alpha\mu\beta\nu}\right] \\
&\quad\times\left[p_\mu p'_\nu + p'_\mu p_\nu - (p\cdot p')g_{\mu\nu} + ip'^\gamma p^\delta\epsilon_{\gamma\mu\delta\nu}\right].
\end{aligned}$$

運動量の積は (A.30) から

$$\begin{aligned}
&[k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu}][p_\mu p'_\nu + p'_\mu p_\nu - (p\cdot p')g_{\mu\nu}] - k'_\alpha k_\beta p'^\gamma p^\delta\epsilon^{\alpha\mu\beta\nu}\epsilon_{\gamma\mu\delta\nu} \\
&= 2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) + 2(p\cdot k)(p'\cdot k') - 2(p\cdot k')(p'\cdot k) \\
&= 4(p\cdot k)(p'\cdot k')
\end{aligned}$$

なので,

$$|\mathcal{M}(\nu u_L)|^2 = 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2(p\cdot k)(p'\cdot k').$$

$\nu + u_R \to \nu + u_R$

不変振幅は

$$i\mathcal{M} = \frac{g^2}{m_Z{}^2\cos^2\theta_w}\frac{-\frac{2}{3}\sin^2\theta_w}{2}\left[\bar{\nu}(k')\gamma_\mu\frac{1-\gamma^5}{2}\nu(k)\right]\left[\bar{u}(p')\gamma^\mu\frac{1+\gamma^5}{2}u(p)\right]$$

なので (A.27) から

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{4}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(-\frac{2}{3}\sin^2\theta_w\right)^2 \mathrm{Tr}\left[\not{k}'\gamma_\mu\frac{1-\gamma^5}{2}\not{k}\gamma_\nu\right]\mathrm{Tr}\left[\not{p}'\gamma^\mu\frac{1+\gamma^5}{2}\not{p}\gamma^\nu\right] \\
&= \left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(-\frac{2}{3}\sin^2\theta_w\right)^2\left[k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu} + ik'_\alpha k_\beta\epsilon^{\alpha\mu\beta\nu}\right] \\
&\quad\times\left[p_\mu p'_\nu + p'_\mu p_\nu - (p\cdot p')g_{\mu\nu} - ip'^\gamma p^\delta\epsilon_{\gamma\mu\delta\nu}\right].
\end{aligned}$$

運動量の積は (A.30) から

$$[k^\mu k'^\nu + k'^\mu k^\nu - (k\cdot k')g^{\mu\nu}][p_\mu p'_\nu + p'_\mu p_\nu - (p\cdot p')g_{\mu\nu}] + k'_\alpha k_\beta p'^\gamma p^\delta\epsilon^{\alpha\mu\beta\nu}\epsilon_{\gamma\mu\delta\nu}$$

$$
\begin{aligned}
&= 2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) - 2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) \\
&= 4(p\cdot k')(p'\cdot k)
\end{aligned}
$$

なので,

$$
|\mathcal{M}(\nu u_R)|^2 = 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(-\frac{2}{3}\sin^2\theta_w\right)^2(p\cdot k')(p'\cdot k).
$$

同様にして

$$
\begin{aligned}
|\mathcal{M}(\nu d_L)|^2 &= 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2(p\cdot k)(p'\cdot k'), \\
|\mathcal{M}(\nu d_R)|^2 &= 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(\frac{1}{3}\sin^2\theta_w\right)^2(p\cdot k')(p'\cdot k).
\end{aligned}
$$

antiquark

$p \leftrightarrow p'$ とすればよいので,

$$
\begin{aligned}
|\mathcal{M}(\nu \bar{u}_L)|^2 &= 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2(p'\cdot k)(p\cdot k'), \\
|\mathcal{M}(\nu \bar{u}_R)|^2 &= 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(-\frac{2}{3}\sin^2\theta_w\right)^2(p'\cdot k')(p\cdot k), \\
|\mathcal{M}(\nu \bar{d}_L)|^2 &= 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2(p'\cdot k)(p\cdot k'), \\
|\mathcal{M}(\nu \bar{d}_R)|^2 &= 4\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left(\frac{1}{3}\sin^2\theta_w\right)^2(p'\cdot k')(p\cdot k).
\end{aligned}
$$

cross section

(17.27) から

$$
y = \frac{\hat{s}+\hat{u}}{\hat{s}} = -\frac{\hat{t}}{\hat{s}}.
$$

(14.1)(14.2)(14.3) と同様に

$$
\frac{d\sigma}{dy} = -\hat{s}\frac{d\sigma}{d\hat{t}} = -2\frac{d\sigma}{d\cos\theta} = -\frac{1}{16\pi\hat{s}}|\mathcal{M}|^2.
$$

up quark の散乱断面積は

$$
\begin{aligned}
\frac{d\sigma(\nu u)}{dy} &= -\frac{1}{16\pi\hat{s}}\frac{|\mathcal{M}(\nu u_L)|^2+|\mathcal{M}(\nu u_R)|^2}{2} \\
&= -\frac{1}{8\pi\hat{s}}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2\frac{\hat{s}^2}{4}+\left(-\frac{2}{3}\sin^2\theta_w\right)^2\frac{\hat{u}^2}{4}\right] \\
&= -\frac{\hat{s}}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2+\left(-\frac{2}{3}\sin^2\theta_w\right)^2(1-y)^2\right] \\
&= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2+\left(-\frac{2}{3}\sin^2\theta_w\right)^2(1-y)^2\right].
\end{aligned}
$$

down quark の散乱断面積は

$$\begin{aligned}\frac{d\sigma(\nu d)}{dy} &= -\frac{1}{16\pi\hat{s}}\frac{|\mathcal{M}(\nu d_L)|^2+|\mathcal{M}(\nu d_R)|^2}{2}\\ &= -\frac{1}{8\pi\hat{s}}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2\frac{\hat{s}^2}{4}+\left(\frac{1}{3}\sin^2\theta_w\right)^2\frac{\hat{u}^2}{4}\right]\\ &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2+\left(\frac{1}{3}\sin^2\theta_w\right)^2(1-y)^2\right].\end{aligned}$$

up antiquark の散乱断面積は

$$\begin{aligned}\frac{d\sigma(\nu\bar{u})}{dy} &= -\frac{1}{16\pi\hat{s}}\frac{|\mathcal{M}(\nu\bar{u}_L)|^2+|\mathcal{M}(\nu\bar{u}_R)|^2}{2}\\ &= -\frac{1}{8\pi\hat{s}}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2\frac{\hat{u}^2}{4}+\left(-\frac{2}{3}\sin^2\theta_w\right)^2\frac{\hat{s}^2}{4}\right]\\ &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2(1-y)^2+\left(-\frac{2}{3}\sin^2\theta_w\right)^2\right].\end{aligned}$$

down antiquark の散乱断面積は

$$\begin{aligned}\frac{d\sigma(\nu\bar{d})}{dy} &= -\frac{1}{16\pi\hat{s}}\frac{|\mathcal{M}(\nu\bar{d}_L)|^2+|\mathcal{M}(\nu\bar{d}_R)|^2}{2}\\ &= -\frac{1}{8\pi\hat{s}}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2\frac{\hat{u}^2}{4}+\left(\frac{1}{3}\sin^2\theta_w\right)^2\frac{\hat{s}^2}{4}\right]\\ &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2(1-y)^2+\left(\frac{1}{3}\sin^2\theta_w\right)^2\right].\end{aligned}$$

$\bar{\nu}+q\to\bar{\nu}+q$ の場合は $k\leftrightarrow k'$ となるので，$\hat{s}^2$ と $\hat{u}^2$ を入れ替えればよい．従って，

$$\begin{aligned}\frac{d\sigma(\bar{\nu}u)}{dy} &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2(1-y)^2+\left(-\frac{2}{3}\sin^2\theta_w\right)^2\right],\\ \frac{d\sigma(\bar{\nu}d)}{dy} &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2(1-y)^2+\left(\frac{1}{3}\sin^2\theta_w\right)^2\right],\\ \frac{d\sigma(\bar{\nu}\bar{u})}{dy} &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2+\left(-\frac{2}{3}\sin^2\theta_w\right)^2(1-y)^2\right],\\ \frac{d\sigma(\bar{\nu}\bar{d})}{dy} &= -\frac{sx}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2+\left(\frac{1}{3}\sin^2\theta_w\right)^2(1-y)^2\right].\end{aligned}$$

なお (20.73)(20.91) から

$$\frac{1}{32\pi}\left(\frac{g^2}{m_Z{}^2\cos^2\theta_w}\right)^2=\frac{G_F{}^2}{\pi}.$$

Chapter 21

Quantization of Spontaneously Broken Gauge Theories

21.1 The R_ξ Gauges

(21.20)

(21.18) の相互作用項のうち，φ を含む項は (21.3)(21.5) より

$$\bar{\psi}_L\phi\psi_R + \bar{\psi}_R\phi^*\psi_L = \bar{\psi}\frac{1+\gamma^5}{2}\psi\frac{i\varphi}{\sqrt{2}} + \bar{\psi}\frac{1-\gamma^5}{2}\psi\frac{-i\varphi}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\varphi(\bar{\psi}\gamma^5\psi).$$

(21.39)

$SU(2)\times U_Y(1)$ 基本表現に属する ϕ の共変微分は (20.60) で与えられる：

$$\begin{aligned}(D_\mu\phi)_i &= \partial_\mu\phi_i - igA^a_\mu(\tau^a)_{ij}\phi_j - ig'B_\mu\frac{1}{2}\phi_i \\ &= \partial_\mu\phi_i + gA^a_\mu\left[-i(\tau^a)_{ij}\right]\phi_j + g'B_\mu\left[-\frac{i}{2}\delta_{ij}\right]\phi_j.\end{aligned}$$

これを (21.33) の形に合わせる．

$$(D_\mu\phi)_i = \partial_\mu\phi_i + gA^a_\mu(T^a)_{ij}\phi_j \quad (a=1,2,3,Y)$$

とすれば $a=1,2,3$ に対しては

$$A^a_\mu = A^a_\mu, \quad T^a = -i\tau^a = T^a.$$

$a=Y$ の場合は[*1]

$$A^Y_\mu = B_\mu, \quad T^Y = -\frac{i}{2}.$$

ここで [20.1.2] の導出と同様にすれば T^Y の変換は

$$T^Y\begin{pmatrix}\phi^1\\ \phi^2\\ \phi^3\\ h\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}-i & \\ & -i\end{pmatrix}\begin{pmatrix}-\phi^2-i\phi^1\\ h+i\phi^3\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}-\phi^1+i\phi^2\\ \phi^3-ih\end{pmatrix} = \begin{pmatrix}-\phi^2\\ \phi^1\\ -h\\ \phi^3\end{pmatrix}$$

[*1] p. 742 の g^2FF^T の計算の後ろに書かれているように $g\to g'$ と解釈する

なので

$$T^Y = \frac{1}{2}\begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}. \tag{21.1.1}$$

$\mathbb{C}^2$ から $\mathbb{R}^4$ に変換して計算する．[20.1.1] から

$$\phi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}$$

である．[20.1.2] を使えば

$$T^1\phi^0 = \frac{1}{2}\begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T^2\phi^0 = \frac{1}{2}\begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix}, \quad T^3\phi^0 = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ v \\ 0 \end{pmatrix}, \quad T^Y\phi^0 = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -v \\ 0 \end{pmatrix}.$$

第 4 成分（真空期待値 v と Higgs 場 h）が 0 なので無視できる．(21.36) の定義から

$$F^a{}_i = T^a_{ij}\phi_{0j} = \frac{v}{2}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ & & -1 \end{pmatrix}, \quad (a = 1, 2, 3, Y; i = 1, 2, 3).$$

$a = Y$ の場合は $g \to g'$ とするので

$$gF^a{}_i = \frac{v}{2}\begin{pmatrix} g & & \\ & g & \\ & & g \\ & & -g' \end{pmatrix}.$$

21.2 The Goldstone Boson Equivalence Theorem

(21.54)

Feynman-'t Hooft gauge ($\xi = 1$) とすれば

$$\contraction{}{A}{{}^a_\mu(k)}{A}A^a_\mu(k)A^b_\nu(q) = \frac{-ig_{\mu\nu}}{k^2 - {m_W}^2}\delta^{ab}(2\pi)^4\,\delta^{(4)}(k+q).$$

(20.63) の定義から

$$\begin{aligned} \contraction{}{W}{{}^+_\mu(k)}{W}W^+_\mu(k)W^-_\nu(q) &= \frac{-ig_{\mu\nu}}{k^2 - {m_W}^2}(2\pi)^4\,\delta^{(4)}(k+q), \\ \contraction{}{W}{{}^+_\mu(k)}{W}W^+_\mu(k)W^+_\nu(q) &= \contraction{}{W}{{}^-_\mu(k)}{W}W^-_\mu(k)W^-_\nu(q) = 0. \end{aligned}$$

(21.56)

ゲージ群 $SU(2) \times U_Y(1)$ の生成子 T^a $(a = 1, 2, 3, Y)$ に対応して ghost は 4 つ存在する．ここで

$$c^+ = \frac{c^1 + ic^2}{\sqrt{2}}, \quad c^- = \frac{c^1 - ic^2}{\sqrt{2}}, \quad c^Z = c^3\cos\theta_w - c^Y\sin\theta_w, \quad c^A = c^3\sin\theta_w + c^Y\cos\theta_w,$$

$$\bar{c}^+ = \frac{\bar{c}^1 - i\bar{c}^2}{\sqrt{2}}, \quad \bar{c}^- = \frac{\bar{c}^1 + i\bar{c}^2}{\sqrt{2}}, \quad \bar{c}^Z = \bar{c}^3 \cos\theta_w - \bar{c}^Y \sin\theta_w, \quad \bar{c}^A = \bar{c}^3 \sin\theta_w + \bar{c}^Y \cos\theta_w$$

とする．Lagragian は (21.52) から

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= \bar{c}^a \left[-(\partial_\mu D^\mu)^{ab} - g^2 (T^a \phi_0) \cdot (T^b \phi)\right] c^b \\ &= \bar{c}^a \left[-\partial^2 \delta^{ac} - g f^{abc} \partial_\mu A^b_\mu - g^2 (T^a \phi_0) \cdot (T^c \phi)\right] c^c.\end{aligned}$$

[20.1.2][21.1.1] から構造定数を求めれば $f^{12Y} = 1$．これを使って計算すれば[*2]

$$\begin{aligned}\overbracket{c^+(k)\bar{c}^+}(q) = \overbracket{c^-(k)\bar{c}^-}(q) &= \frac{i}{k^2 - {m_W}^2}(2\pi)^4\,\delta^{(4)}(k-q), \\ \overbracket{c^Z(k)\bar{c}^Z}(q) &= \frac{i}{k^2 - {m_Z}^2}(2\pi)^4\,\delta^{(4)}(k-q), \\ \overbracket{c^A(k)\bar{c}^A}(q) &= \frac{i}{k^2}(2\pi)^4\,\delta^{(4)}(k-q).\end{aligned} \tag{21.2.2}$$

Fig 21.8: Feynman rules of Goldstone bosons

Goldstone boson の propagator は (21.55) から

$$\overbracket{\phi_i(k)\phi_j}(q) = \frac{i}{k^2 - m^2}\delta_{ij}(2\pi)^4\,\delta^{(4)}(k+q).$$

(21.79) の定義から

$$\begin{aligned}\overbracket{\phi_+(k)\phi_-}(q) &= \frac{i}{k^2 - {m_W}^2}(2\pi)^4\,\delta^{(4)}(k+q), \\ \overbracket{\phi_+(k)\phi_+}(q) &= \overbracket{\phi_-(k)\phi_-}(q) = 0.\end{aligned}$$

(21.38)(21.79) から $SU(2)$ scalar 場は

$$\phi = \begin{pmatrix} -i\phi^- \\ (v + h + i\phi^3)/\sqrt{2} \end{pmatrix}, \quad \phi^\dagger = \begin{pmatrix} i\phi^+ & \dfrac{v + h - i\phi^3}{\sqrt{2}} \end{pmatrix}$$

となる．

(20.60) に (20.65)(20.66) の結果を適用 ($Y = 1/2$) して

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{1}{\sqrt{g^2 + g'^2}} Z^0_\mu (g^2 T^3 - g'^2/2) - i\frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu (T^3 + 1/2).$$

(20.68) を代入して

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{1}{\sqrt{g^2 + g'^2}} Z^0_\mu (g^2 T^3 - g'^2/2) - ieA_\mu (T^3 + 1/2).$$

ϕ の共変微分は

$$D_\mu \phi = \partial_\mu \phi - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-)\phi - i\frac{1}{\sqrt{g^2 + g'^2}} Z^0_\mu (g^2 T^3 - g'^2/2)\phi - ieA_\mu (T^3 + 1/2)\phi$$

[*2] `./src/py/ghost.ipynb`

$$
\begin{aligned}
&= \partial_\mu \phi - i\frac{g}{\sqrt{2}} \begin{pmatrix} & W_\mu^+ \\ W_\mu^- & \end{pmatrix} \phi - \frac{i}{2} Z_\mu^0 \begin{pmatrix} \frac{g^2-g'^2}{\sqrt{g^2+g'^2}} & \\ & -\sqrt{g^2+g'^2} \end{pmatrix} \phi - ieA_\mu \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \phi \\
&= \begin{pmatrix} -i\partial_\mu \phi^- \\ \partial_\mu (h+i\phi^3)/\sqrt{2} \end{pmatrix} - i\frac{g}{\sqrt{2}} \begin{pmatrix} W_\mu^+ (v+h+i\phi^3)/\sqrt{2} \\ -iW_\mu^- \phi^- \end{pmatrix} \\
&\quad - \frac{i}{2} Z_\mu^0 \begin{pmatrix} -i\frac{g^2-g'^2}{\sqrt{g^2+g'^2}} \phi^- \\ -\sqrt{g^2+g'^2}(v+h+i\phi^3)/\sqrt{2} \end{pmatrix} - ieA_\mu \begin{pmatrix} -i\phi^- \\ 0 \end{pmatrix}.
\end{aligned}
$$

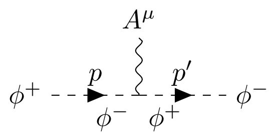
(20.70) より

$$
\begin{aligned}
(D_\mu \phi)_1 &= -i\partial_\mu \phi^- - i\frac{g}{2} W_\mu^+ (v+h+i\phi^3) - \frac{1}{2}\frac{g^2-g'^2}{\sqrt{g^2+g'^2}} Z_\mu^0 \phi^- - eA_\mu \phi^- \\
&= -i\partial_\mu \phi^- - i\frac{g}{2} W_\mu^+ (v+h+i\phi^3) - \frac{g}{\cos\theta_w}\left(\frac{1}{2} - \sin^2\theta_w\right) Z_\mu^0 \phi^- - eA_\mu \phi^-, \\
(D_\mu \phi)_2 &= \frac{1}{\sqrt{2}} \partial_\mu (h+i\phi^3) - \frac{g}{\sqrt{2}} g W_\mu^- \phi^- + \frac{i}{2\sqrt{2}} \sqrt{g^2+g'^2} Z_\mu^0 (v+h+i\phi^3) \\
&= \frac{1}{\sqrt{2}} \partial_\mu (h+i\phi^3) - \frac{g}{\sqrt{2}} g W_\mu^- \phi^- + \frac{i}{2\sqrt{2}} \frac{g}{\cos\theta_w} Z_\mu^0 (v+h+i\phi^3).
\end{aligned}
$$

(20.111) の Lagrangian は (20.63)(20.112)(20.114) から

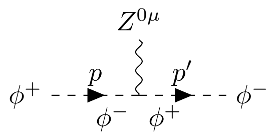
$$
\begin{aligned}
\mathcal{L}_{\text{Higgs}} &= |D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger\phi)^2 \\
&= (D_\mu\phi)_1 (D^\mu\phi)_1^\dagger + (D_\mu\phi)_2 (D^\mu\phi)_2^\dagger + \frac{{m_h}^2}{2}\phi^\dagger\phi - \frac{g^2}{8}\frac{{m_h}^2}{{m_W}^2}(\phi^\dagger\phi)^2
\end{aligned}
\qquad [21.2.3]
$$

で与えられる．



頂点の ϕ^- には運動量 p が入る ($e^{-ip\cdot x}$) ので，$\partial^\mu \phi^- \to -ip^\mu \phi^-$．よって $\phi^-\phi^+A$ 頂点は

$$
\begin{aligned}
i\mathcal{L}_{\text{Higgs}} &= i(-i\partial^\mu \phi^-)(-eA_\mu \phi^+) + i(i\partial^\mu\phi^+)(-eA_\mu\phi^-) + \cdots \\
&\to i(-p^\mu)(-e) + i(-p'^\mu)(-e) \\
&= ie(p+p')^\mu.
\end{aligned}
\qquad [21.2.4]
$$



頂点の ϕ^- には運動量 p が入る ($e^{-ip\cdot x}$) ので，$\partial^\mu \phi^- \to -ip^\mu \phi^-$．よって $\phi^-\phi^+Z^0$ 頂点は

$$
\begin{aligned}
i\mathcal{L}_{\text{Higgs}} &= i(-i\partial^\mu \phi^-)\frac{-g}{\cos\theta_w}\left(\frac{1}{2}-\sin^2\theta_w\right) Z_\mu^0 \phi^+ + i(i\partial^\mu \phi^+)\frac{-g}{\cos\theta_w}\left(\frac{1}{2}-\sin^2\theta_w\right) Z_\mu^0 \phi^- + \cdots \\
&\to i(-p^\mu)\frac{-g}{\cos\theta_w}\left(\frac{1}{2}-\sin^2\theta_w\right) + i(-p'^\mu)\frac{-g}{\cos\theta_w}\left(\frac{1}{2}-\sin^2\theta_w\right) \\
&= \frac{ig(p+p')^\mu}{\cos\theta_w}\left(\frac{1}{2}-\sin^2\theta_w\right).
\end{aligned}
$$

(21.108)

`./src/py/problem_21_2_LL.ipynb` から

$$\begin{aligned}
i\mathcal{M}(e_L^- e_R^+ \to W_L^+ W_L^-) &= \frac{ie^2\left(-2E^2 - m_W{}^2\right)\sqrt{E^2 - m_W{}^2}\sin\theta}{E\left(4E^2\cos^2\theta_w - m_W{}^2\right)} \\
&\quad + \frac{2iEe^2\sqrt{E^2 - m_W{}^2}\left(2E^2 + m_W{}^2\right)\sin\theta}{m_W{}^2\left(2E^2\cos 2\theta_w + 2E^2 - m_W{}^2\right)\tan^2\theta_w} \\
&\quad + \frac{iEe^2\left(2E^3\cos\theta - 2E^2\sqrt{E^2 - m_W{}^2} - m_W{}^2\sqrt{E^2 - m_W{}^2}\right)\sin\theta}{m_W{}^2\left(2E^2 - 2E\sqrt{E^2 - m_W{}^2}\cos\theta - m_W{}^2\right)\sin^2\theta_w} \\
&= \frac{i\beta e^2\left(-2E^2 - m_W{}^2\right)\sin\theta}{4E^2\cos^2\theta_w - m_W{}^2} + \frac{2i\beta E^2 e^2\left(2E^2 + m_W{}^2\right)\sin\theta}{m_W{}^2\left(4E^2\cos^2\theta_w - m_W{}^2\right)\tan^2\theta_w} \\
&\quad + \frac{iE^2e^2\left(2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta\right)\sin\theta}{m_W{}^2\left(2E^2 - 2\beta E^2\cos\theta - m_W{}^2\right)\sin^2\theta_w}
\end{aligned}$$

(20.73) から

$$\begin{aligned}
&= -\frac{i\beta e^2\left(2E^2 + m_W{}^2\right)m_Z{}^2\sin\theta}{m_W{}^2(4E^2 - m_Z{}^2)} + \frac{2i\beta E^2e^2\left(2E^2 + m_W{}^2\right)\sin\theta}{m_W{}^2\left(4E^2 - m_Z{}^2\right)\sin^2\theta_w} \\
&\quad + \frac{iE^2e^2\left(2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta\right)\sin\theta}{m_W{}^2\left(2E^2 - 2\beta E^2\cos\theta - m_W{}^2\right)\sin^2\theta_w} \\
&= -i\beta e^2\sin\theta\frac{m_Z{}^2}{m_W{}^2}\frac{2E^2 + m_W{}^2}{4E^2 - m_Z{}^2} + i\beta e^2\sin\theta\frac{1}{2\sin^2\theta_w}\frac{4E^2}{4E^2 - m_Z{}^2}\frac{m_W{}^2 + 2E^2}{m_W{}^2} \\
&\quad + \frac{ie^2\left(2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta\right)\sin\theta}{m_W{}^2\left(1 + \beta^2 - 2\beta\cos\theta\right)\sin^2\theta_w} \\
&= -i\beta e^2\sin\theta\frac{m_Z{}^2}{m_W{}^2}\frac{2E^2 + m_W{}^2}{4E^2 - m_Z{}^2} + i\beta e^2\sin\theta\frac{1}{2\sin^2\theta_w}\frac{4E^2}{4E^2 - m_Z{}^2}\left(1 + \frac{m_Z{}^2}{2m_W{}^2} + \frac{4E^2 - m_Z{}^2}{2m_W{}^2}\right) \\
&\quad + \frac{ie^2\left(2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta\right)\sin\theta}{m_W{}^2\left(1 + \beta^2 - 2\beta\cos\theta\right)\sin^2\theta_w} \\
&= -i\beta e^2\sin\theta\frac{m_Z{}^2}{m_W{}^2}\frac{2E^2 + m_W{}^2}{4E^2 - m_Z{}^2} + i\beta e^2\sin\theta\frac{1}{2\sin^2\theta_w}\frac{4E^2}{4E^2 - m_Z{}^2}\left(1 + \frac{m_Z{}^2}{2m_W{}^2}\right) \\
&\quad + i\beta e^2\sin\theta\frac{1}{\sin^2\theta_w}\frac{E^2}{m_W{}^2} + \frac{ie^2\left(2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta\right)\sin\theta}{m_W{}^2\left(1 + \beta^2 - 2\beta\cos\theta\right)\sin^2\theta_w} \\
&= -i\beta e^2\sin\theta\frac{m_Z{}^2}{m_W{}^2}\frac{s/2 + m_W{}^2}{s - m_Z{}^2} + i\beta e^2\sin\theta\frac{1}{2\sin^2\theta_w}\frac{s}{s - m_Z{}^2}\left(1 + \frac{m_Z{}^2}{2m_W{}^2}\right) \\
&\quad + i\beta e^2\sin\theta\frac{1}{2\sin^2\theta_w}2\left[\frac{E^2}{m_W{}^2} + \frac{1}{\beta m_W{}^2}\frac{2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta}{1 + \beta^2 - 2\beta\cos\theta}\right].
\end{aligned}$$

第 3 項は

$$\begin{aligned}
[\cdots] &= \frac{E^2}{m_W{}^2} + \frac{1}{\beta m_W{}^2}\frac{2E^2\cos\theta - 2\beta E^2 - m_W{}^2\beta}{1 + \beta^2 - 2\beta\cos\theta} \\
&= \frac{\beta E^2(1 + \beta^2 - 2\beta\cos\theta) + (2E^2\cos\theta - 2\beta E^2 - \beta m_W{}^2)}{\beta m_W{}^2(1 + \beta^2 - 2\beta\cos\theta)} \\
&= \frac{\beta E^2 + \beta^3E^2 - 2\beta^2E^2\cos\theta + 2E^2\cos\theta - 2\beta E^2 - \beta m_W{}^2}{\beta m_W{}^2(1 + \beta^2 - 2\beta\cos\theta)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\beta^3 E^2 - \beta E^2 + 2(1-\beta^2)E^2\cos\theta - \beta {m_W}^2}{\beta {m_W}^2(1+\beta^2-2\beta\cos\theta)} \\
&= \frac{\beta^3 E^2 - \beta E^2 + 2{m_W}^2\cos\theta - \beta {m_W}^2}{\beta {m_W}^2(1+\beta^2-2\beta\cos\theta)} \\
&= \frac{-\beta(1-\beta^2)E^2 + {m_W}^2/\beta - (1+\beta^2-2\beta\cos\theta){m_W}^2/\beta}{\beta {m_W}^2(1+\beta^2-2\beta\cos\theta)} \\
&= \frac{-\beta {m_W}^2 + {m_W}^2/\beta}{\beta {m_W}^2(1+\beta^2-2\beta\cos\theta)} - \frac{1}{\beta^2} \\
&= \frac{1-\beta^2}{\beta^2(1+\beta^2-2\beta\cos\theta)} - \frac{1}{\beta^2} \\
&= \frac{{m_W}^2}{E^2\beta^2(1+\beta^2-2\beta\cos\theta)} - \frac{1}{\beta^2} \\
&= \frac{4{m_W}^2}{s\beta^2(1+\beta^2-2\beta\cos\theta)} - \frac{1}{\beta^2}.
\end{aligned}
$$

さらに

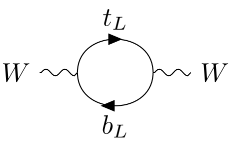
$$ie^2\bar{v}_L(\not{k}_+ - \not{k}_-)u_L\frac{1}{s} = -i\beta e^2\sin\theta$$

を併せれば，右辺を得る．

21.3 One-Loop Corrections to the Weak-Interaction Gauge Theory

(21.153)

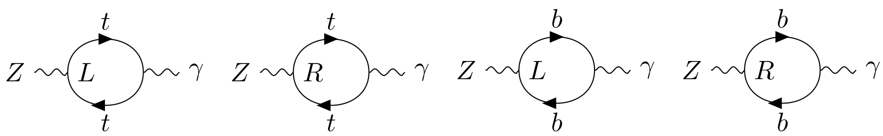
$\Pi_{WW} = g^2\Pi_{11}$ は



で与えられる．(20.79)(20.80)(21.150) から

$$\Pi_{11}({q_X}^2) = -\frac{3}{2}\frac{4}{(4\pi)^2}\left[\left(\frac{{q_X}^2}{6} - \frac{{m_t}^2+{m_b}^2}{4}\right)E - {q_X}^2 b_2(btX) + \frac{{m_t}^2 b_1(btX) + {m_b}^2 b_1(tbX)}{2}\right].$$

$\Pi_{\gamma Z} = (eg/\cos\theta_w)[\Pi_{3Q} - \sin^2\theta_w\Pi_{QQ}]$ は



で与えられる．(20.79)(20.80) から (t,t,L) の diagram は

$$\frac{2}{3}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w\right) \times (\Pi_{LL} + \Pi_{LR})$$

となる．(21.149)(21.150)(21.151) から，全ての diagram の合計は

$$\Pi_{3Q} - \sin^2\theta_w\Pi_{QQ} = \frac{2}{3}\left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w - \frac{2}{3}\sin^2\theta_w\right)3[\Pi_{LL}(ttX) + \Pi_{LR}(ttX)]$$

$$
\begin{aligned}
&-\frac{1}{3}\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w+\frac{1}{3}\sin^2\theta_w\right)3[\Pi_{LL}(bbX)+\Pi_{LR}(bbX)]\\
&=-\frac{12}{(4\pi)^2}\left(\frac{1}{3}-\frac{8}{9}\sin^2\theta_w\right)\left[\frac{{q_X}^2}{6}E-{q_X}^2b_2(ttX)\right]\\
&\quad-\frac{12}{(4\pi)^2}\left(\frac{1}{6}-\frac{2}{9}\sin^2\theta_w\right)\left[\frac{{q_X}^2}{6}E-{q_X}^2b_2(bbX)\right]\\
&=-\frac{6}{(4\pi)^2}\left[\frac{{q_X}^2}{6}E-\frac{2}{3}{q_X}^2b_2(ttX)-\frac{1}{3}{q_X}^2b_2(bbX)\right]-\sin^2\theta_w\Pi_{QQ}.
\end{aligned}
$$

よって

$$
\Pi_{3Q}({q_X}^2)=-\frac{6}{(4\pi)^2}\left[\frac{{q_X}^2}{6}E-\frac{2}{3}{q_X}^2b_2(ttX)-\frac{1}{3}{q_X}^2b_2(bbX)\right].
$$

$\Pi_{ZZ}=(g/\cos\theta_w)^2[\Pi_{33}-2\sin^2\theta_w\Pi_{3Q}+\sin^4\theta_w\Pi_{QQ}]$ は

$$
\begin{aligned}
&\Pi_{33}-2\sin^2\theta_w\Pi_{3Q}+\sin^4\theta_w\Pi_{QQ}\\
&=\left[\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)^2+\left(-\frac{2}{3}\sin^2\theta_w\right)^2\right]3\Pi_{LL}(ttX)\\
&\quad+2\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)\left(-\frac{2}{3}\sin^2\theta_w\right)3\Pi_{LR}(ttX)\\
&\quad+\left[\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)^2+\left(\frac{1}{3}\sin^2\theta_w\right)^2\right]3\Pi_{LL}(bbX)\\
&\quad+2\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)\left(\frac{1}{3}\sin^2\theta_w\right)3\Pi_{LR}(bbX)\\
&=-\frac{12}{(4\pi)^2}\left(\frac{1}{4}-\frac{2}{3}\sin^2\theta_w+\frac{8}{9}\sin^4\theta_w\right)\left[\left(\frac{{q_X}^2}{6}-\frac{{m_t}^2}{2}\right)E-{q_X}^2b_2(ttX)+{m_t}^2b_1(ttX)\right]\\
&\quad-\frac{12}{(4\pi)^2}\left(-\frac{1}{3}\sin^2\theta_w+\frac{4}{9}\sin^4\theta_w\right)[{m_t}^2E-2{m_t}^2b_1(ttX)]\\
&\quad-\frac{12}{(4\pi)^2}\left(\frac{1}{4}-\frac{1}{3}\sin^2\theta_w+\frac{2}{9}\sin^4\theta_w\right)\left[\left(\frac{{q_X}^2}{6}-\frac{{m_b}^2}{2}\right)E-{q_X}^2b_2(bbX)+{m_b}^2b_1(bbX)\right]\\
&\quad-\frac{12}{(4\pi)^2}\left(-\frac{1}{6}\sin^2\theta_w+\frac{1}{9}\sin^4\theta_w\right)[{m_b}^2E-2{m_b}^2b_1(bbX)]\\
&=-\frac{6}{(4\pi)^2}\left[\left(\frac{{q_X}^2}{6}-\frac{{m_t}^2+{m_b}^2}{4}\right)E-{q_X}^2\frac{b_2(ttX)+b_2(bbX)}{2}+\frac{{m_t}^2b_1(ttX)+{m_b}^2b_1(bbX)}{2}\right]\\
&\quad-2\sin^2\theta_w\Pi_{3Q}+\sin^4\theta_w\Pi_{QQ}.
\end{aligned}
$$

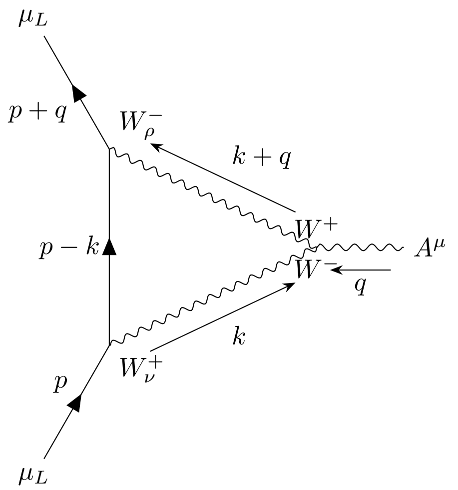
よって

$$
\Pi_{33}({q_X}^2)=\frac{-6}{(4\pi)^2}\left[\left(\frac{{q_X}^2}{6}-\frac{{m_t}^2+{m_b}^2}{4}\right)E-{q_X}^2\frac{b_2(ttX)+b_2(bbX)}{2}+\frac{{m_t}^2b_1(ttX)+{m_b}^2b_1(bbX)}{2}\right].
$$

Problems

Problem 21.1: Weak interaction contribution to the muon $g-2$

WW diagram



頂点の 1 loop 補正は

$$
\begin{aligned}
&\bar{u}(p+q)(-ie\delta\Gamma^\mu - ie\delta\Gamma_5^\mu)u(p) \\
&= \int \frac{d^dk}{(2\pi)^d}\bar{u}(p+q)\frac{ig}{\sqrt{2}}\gamma_\rho\frac{i}{\not{p}-\not{k}}\frac{ig}{\sqrt{2}}\gamma_\nu\frac{1-\gamma^5}{2}u(p)\frac{-i}{k^2-m_W{}^2}\frac{-i}{(k+q)^2-m_W{}^2} \\
&\quad\times ie[g^{\nu\rho}(2k+q)^\mu + g^{\mu\nu}(-k+q)^\rho + g^{\mu\rho}(-k-2q)^\nu] \\
&= -\frac{eg^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)} \\
&\quad\times \bar{u}(p+q)\left[\gamma^\nu(\not{p}-\not{k})\gamma_\nu(2k+q)^\mu + (\not{k}-\not{q})(\not{k}-\not{p})\gamma^\mu + \gamma^\mu(\not{k}-\not{p})(\not{k}+2\not{q})\right]\frac{1-\gamma^5}{2}u(p).
\end{aligned}
$$

γ^5 を含まない項は

$$
\begin{aligned}
\delta\Gamma^\mu = -i\frac{g^2}{4}\int\frac{d^dk}{(2\pi)^d}&\frac{1}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)} \\
&\times\left[\gamma^\nu(\not{p}-\not{k})\gamma_\nu(2k+q)^\mu + (\not{k}-\not{q})(\not{k}-\not{p})\gamma^\mu + \gamma^\mu(\not{k}-\not{p})(\not{k}+2\not{q})\right].
\end{aligned}
$$

分母は (A.39) から

$$
-\frac{ig^2}{2}\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(1-x-y-z)\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3}.
$$

ただし，

$$
k = \ell + (zp - yq), \quad \Delta = -y(1-y)q^2 - z(1-z)p^2 - 2yzp\cdot q + (1-z)m_W{}^2.
$$

ここで $p^2 = m^2$,

$$
2p\cdot q = (p+q)^2 - p^2 - q^2 = m^2 - m^2 - q^2 = -q^2 \tag{21.3.5}
$$

なので

$$\begin{aligned}\Delta &= -y(1-y)q^2 - z(1-z)m^2 + yzq^2 + (1-z){m_W}^2 \\ &= -y(1-y-z)q^2 - z(1-z)m^2 + (1-z){m_W}^2 \\ &= -xyq^2 - z(1-z)m^2 + (1-z){m_W}^2.\end{aligned}$$

Dirac 方程式から

$$\not{p}u(p) = mu(p), \quad \bar{u}(p+q)(\not{p}+\not{q}) = \bar{u}(p+q)m.$$

分子は (A.55) から

$$\begin{aligned}A_{WW} &= \gamma^\nu(\not{p}-\not{k})\gamma_\nu(2k+q)^\mu + (\not{k}-\not{q})(\not{k}-\not{p})\gamma^\mu + \gamma^\mu(\not{k}-\not{p})(\not{k}+2\not{q}) \\ &= (d-2)(\not{k}-\not{p})(2k+q)^\mu + (\not{k}-\not{q})(\not{k}-\not{p})\gamma^\mu + \gamma^\mu(\not{k}-\not{p})(\not{k}+2\not{q}).\end{aligned}$$

$k = \ell + (zp - yq)$ を代入すると，(A.44)(A.45) より分子の ℓ^2 の項は発散し，ℓ^0 の定数項 A^0_{WW} が収束する．

第 1 項の定数項は

$$\begin{aligned}& -2\left[(1-z)\not{p} + y\not{q}\right]\left[2zp^\mu + (1-2y)q^\mu\right] \\ &\sim -2(1-z)m\left[2zp^\mu + (1-2y)q^\mu\right] \\ &= -2(1-z)m\left[z(p^\mu + p'^\mu) + (1-2y-z)q^\mu\right].\end{aligned}$$

第 2 項の定数項は

$$\begin{aligned}& -\left[z\not{p} - (1+y)\not{q}\right]\left[(1-z)\not{p} + y\not{q}\right]\gamma^\mu \\ &\sim -\left[z(m-\not{q}) - (1+y)\not{q}\right]\left[(1-z)\not{p}\gamma^\mu + y\not{q}\gamma^\mu\right] \\ &\sim -\left[zm - (1+y+z)\not{q}\right]\left[(1-z)(2p^\mu - \gamma^\mu\not{p}) + y\not{q}\gamma^\mu\right] \\ &\sim -\left[zm - (1+y+z)\not{q}\right]\left[2(1-z)p^\mu - (1-z)m\gamma^\mu + y\not{q}\gamma^\mu\right] \\ &= -2z(1-z)mp^\mu + z(1-z)m^2\gamma^\mu - yzm\not{q}\gamma^\mu \\ &\quad + 2(1-z)(1+y+z)p^\mu\not{q} - (1-z)(1+y+z)m\not{q}\gamma^\mu + y(1+y+z)q^2\gamma^\mu \\ &\sim [y(1+y+z)q^2 + z(1-z)m^2]\gamma^\mu - 2z(1-z)mp^\mu - [(1-z)(1+y+z) + yz]m\not{q}\gamma^\mu \\ &= [y(1+y+z)q^2 + z(1-z)m^2]\gamma^\mu - 2z(1-z)mp^\mu - (1+y-z^2)m\not{q}\gamma^\mu.\end{aligned}$$

ここで

$$\not{q}\gamma^\mu \sim (m-\not{p})\gamma^\mu = m\gamma^\mu - \not{p}\gamma^\mu = m\gamma^\mu - (2p^\mu - \gamma^\mu\not{p}) \sim m\gamma^\mu - (2p^\mu - \gamma^\mu m) = 2m\gamma^\mu - 2p^\mu$$

なので

$$\begin{aligned}&\sim [y(1+y+z)q^2 + z(1-z)m^2]\gamma^\mu - 2z(1-z)mp^\mu - (1+y-z^2)m(2m\gamma^\mu - 2p^\mu) \\ &= [y(1+y+z)q^2 - (2+2y-z-z^2)m^2]\gamma^\mu + 2(1+y-z)mp^\mu \\ &= [y(1+y+z)q^2 - (2+2y-z-z^2)m^2]\gamma^\mu + (1+y-z)m(p^\mu + p'^\mu) - (1+y-z)mq^\mu.\end{aligned}$$

第 3 項の定数項は

$$\begin{aligned}& -\gamma^\mu\left[(1-z)\not{p} + y\not{q}\right]\left[z\not{p} + (2-y)\not{q}\right] \\ &= -\gamma^\mu\left[z(1-z)\not{p}^2 + y(2-y)\not{q}^2 + yz\not{q}\not{p} + (1-z)(2-y)\not{p}\not{q}\right] \\ &\sim -\gamma^\mu\left[z(1-z)m^2 + y(2-y)q^2 + yz\not{q}\not{p} + (2-y-2z+yz)\not{p}\not{q}\right]\end{aligned}$$

$$
\begin{aligned}
&= -\gamma^\mu \left[z(1-z)m^2 + y(2-y)q^2 + yz(\not{q}\not{p} + \not{q}\not{p}) + (2-y-2z)\not{p}\not{q}\right] \\
&= -\gamma^\mu \left[z(1-z)m^2 + y(2-y)q^2 + yz(2p\cdot q) + (2-y-2z)\not{p}\not{q}\right] \\
&= \left[-z(1-z)m^2 - y(2-y)q^2\right]\gamma^\mu - yz(2p\cdot q)\gamma^\mu - (2-y-2z)\gamma^\mu\not{p}\not{q}.
\end{aligned}
$$

[21.3.5] および

$$
\begin{aligned}
\gamma^\mu\not{p}\not{q} &= (2p^\mu - \not{p}\gamma^\mu)\not{q} \sim -\not{p}\gamma^\mu\not{q} \sim (\not{q}-m)\gamma^\mu\not{q} = (\not{q}-m)(2q^\mu - \not{q}\gamma^\mu) \\
&= 2q^\mu\not{q} - 2mq^\mu - \not{q}\not{q}\gamma^\mu + m\not{q}\gamma^\mu \sim -2mq^\mu - q^2\gamma^\mu + m(2m\gamma^\mu - 2p^\mu) \\
&= (2m^2 - q^2)\gamma^\mu - 2mp'^\mu
\end{aligned}
$$

から

$$
\begin{aligned}
&\sim \left[-z(1-z)m^2 - y(2-y)q^2\right]\gamma^\mu + yzq^2\gamma^\mu - (2-y-2z)[(2m^2-q^2)\gamma^\mu - 2mp'^\mu] \\
&= \left[(-4+2y+3z+z^2)m^2 + (2-3y-2z+y^2+yz)q^2\right]\gamma^\mu + 2(2-y-2z)mp'^\mu \\
&= \left[(-4+2y+3z+z^2)m^2 + (2-3y-2z+y^2+yz)q^2\right]\gamma^\mu \\
&\qquad + (2-y-2z)m(p^\mu + p'^\mu) + (2-y-2z)mq^\mu.
\end{aligned}
$$

以上を全て足して,

$$
\begin{aligned}
A^0_{WW} &\sim -2(1-z)m\left[z(p^\mu+p'^\mu) + (1-2y-z)q^\mu\right] \\
&\quad + [y(1+y+z)q^2 - (2+2y-z-z^2)m^2]\gamma^\mu + (1+y-z)m(p^\mu+p'^\mu) - (1+y-z)mq^\mu \\
&\quad + \left[(-4+2y+3z+z^2)m^2 + (2-3y-2z+y^2+yz)q^2\right]\gamma^\mu \\
&\qquad + (2-y-2z)m(p^\mu+p'^\mu) + (2-y-2z)mq^\mu \\
&= \left[-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2\right]\gamma^\mu \\
&\qquad + (1-z)(3-2z)m(p^\mu+p'^\mu) - (2z-1)(1-2y-z)mq^\mu \\
&= \left[-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2\right]\gamma^\mu \\
&\qquad + (1-z)(3-2z)m(p^\mu+p'^\mu) - (2z-1)(x-y)mq^\mu.
\end{aligned}
$$

被積分函数の分母は $x \leftrightarrow y$ について対称なので,

$$
A^0_{WW} \to \left[-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2\right]\gamma^\mu + (1-z)(3-2z)m(p^\mu+p'^\mu). \tag{21.3.6}
$$

Feynman rules of Goldstone bosons

(20.98) から lepton, neutrino と $SU(2)$ scalar の相互作用は

$$
\Delta\mathcal{L}_e = -\lambda_e \bar{E}_L \phi e_R + (\text{h.c})
$$

(21.38)(21.79)(20.75) を代入して

$$
\begin{aligned}
-\lambda_e \bar{E}_L \phi e_R &= -\lambda_e \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix} \begin{pmatrix} -i\phi^- \\ (v+h+i\phi^3)/\sqrt{2}\end{pmatrix} e_R \\
&= i\lambda_e \bar{\nu}_L e_R \phi^- - \frac{\lambda_e}{\sqrt{2}} \bar{e}_L e_R (v+h+i\phi^3).
\end{aligned}
$$

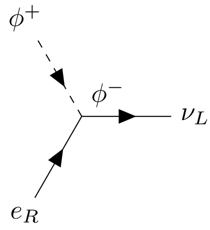
(20.63)(20.100) から

$$
\lambda_e = \frac{g}{\sqrt{2}}\frac{m_e}{m_W}.
$$

よって

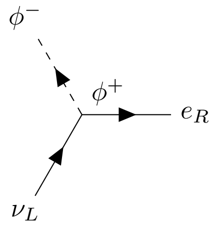
$$\Delta\mathcal{L}_e = i\frac{g}{\sqrt{2}}\frac{m_e}{m_W}\bar{\nu}_L e_R \phi^- - \frac{g}{2}\frac{m_e}{m_W}\bar{e}_L e_R(v + h + i\phi^3) + (\text{h.c}). \tag{21.3.7}$$

$e\nu\phi^-$ の頂点は



$$= -\frac{g}{\sqrt{2}}\frac{m_e}{m_W}\frac{1+\gamma^5}{2}.$$

$e\nu\phi^+$ の頂点は



$$= \frac{g}{\sqrt{2}}\frac{m_e}{m_W}\frac{1-\gamma^5}{2}.$$

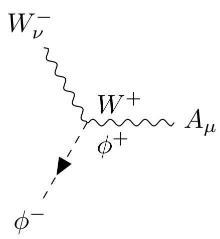
[21.2.3] から

$$\mathcal{L}_{\text{Higgs}} = ie\frac{gv}{2}A^\mu\phi^+W^+_\mu - ie\frac{gv}{2}A^\mu\phi^-W^-_\mu.$$

(20.63) を代入して

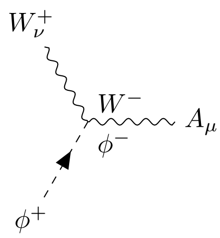
$$i\mathcal{L}_{\text{Higgs}} = -em_W A^\mu\phi^+W^+_\mu + em_W A^\mu\phi^-W^-_\mu.$$

$A\phi^+W^+$ の頂点は



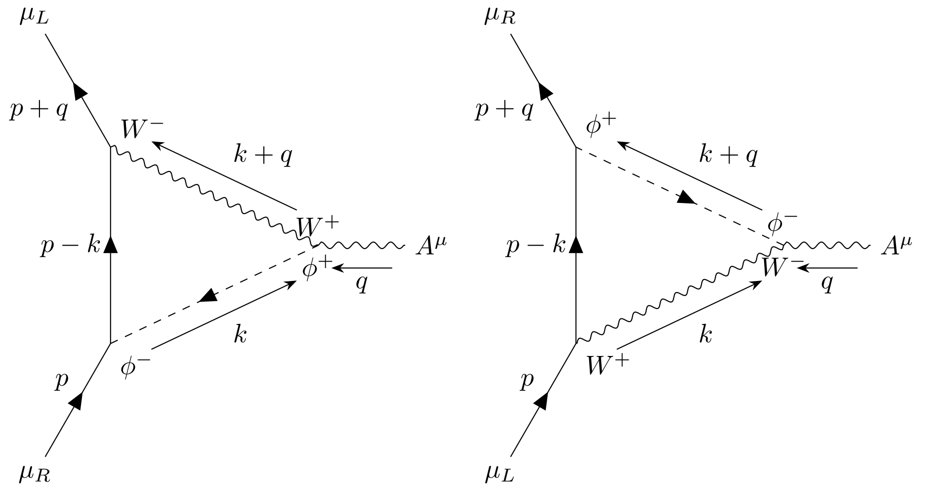
$$= -em_W g^{\mu\nu}. \tag{21.3.8}$$

$A\phi^-W^-$ の頂点は



$$= em_W g^{\mu\nu}. \tag{21.3.9}$$

$\phi W + W\phi$ diagrams



頂点の 1 loop 補正は

$$\begin{aligned}
&\bar{u}(p+q)(-ie\delta\Gamma^\mu - ie\delta\Gamma_5^\mu)u(p) \\
&= \int \frac{d^dk}{(2\pi)^d}\bar{u}(p+q)\frac{ig}{\sqrt{2}}\gamma^\mu \frac{i}{\not{p}-\not{k}}\frac{-g}{\sqrt{2}}\frac{m}{m_W}\frac{1+\gamma^5}{2}u(p)\frac{-i}{(k+q)^2-m_W{}^2}\frac{i}{k^2-m_W{}^2}(-em_W) \\
&+ \int \frac{d^dk}{(2\pi)^d}\bar{u}(p+q)\frac{g}{\sqrt{2}}\frac{m}{m_W}\frac{1+\gamma^5}{2}\frac{i}{\not{p}-\not{k}}\frac{ig}{\sqrt{2}}\gamma^\mu u(p)\frac{i}{(k+q)^2-m_W{}^2}\frac{-i}{k^2-m_W{}^2}em_W \\
&= -\frac{eg^2m_\mu}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)} \\
&\quad \times \bar{u}(p+q)\left[\gamma^\mu(\not{k}-\not{p})\frac{1+\gamma^5}{2}+\frac{1+\gamma^5}{2}(\not{k}-\not{p})\gamma^\mu\right]u(p).
\end{aligned}$$

γ^5 を含まない項は

$$\begin{aligned}
\delta\Gamma^\mu &= -\frac{ig^2}{4}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)}m\left[\gamma^\mu(\not{k}-\not{p})+(\not{k}-\not{p})\gamma^\mu\right] \\
&= -\frac{ig^2}{4}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)}2m\left[k^\mu-p^\mu\right].
\end{aligned}$$

分子の定数項は

$$\begin{aligned}
A^0_{\phi W} &= 2m\left[-(1-z)p^\mu - yq^\mu\right] \\
&= -2(1-z)mp^\mu - 2ymq^\mu \\
&= -(1-z)m(p^\mu+p'^\mu) + (1-2y-z)mq^\mu \\
&= -(1-z)m(p^\mu+p'^\mu) + (z-y)q^\mu \\
&\to -(1-z)m(p^\mu+p'^\mu).
\end{aligned}$$

[21.3.6] と併せて

$$\begin{aligned}
A^0_{WW} + A^0_{\phi W} &= \left[-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2\right]\gamma^\mu \\
&\quad + (1-z)(3-2z)m(p^\mu+p'^\mu) - (1-z)m(p^\mu+p'^\mu)
\end{aligned}$$

$$
\begin{aligned}
&= \left[-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2\right]\gamma^\mu \\
&\quad + 2(1-z)(2-z)m(p^\mu + p'^\mu).
\end{aligned}
$$

Gordon 恒等式 (6.32) を使えば

$$
= [\cdots]\gamma^\mu - 4(1-z)(2-z)m^2 \frac{i\sigma^{\mu\nu}q_\nu}{2m}.
$$

(6.33) から

$$
\begin{aligned}
F_2(q^2) &= \frac{ig^2}{2}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z) \int \frac{d^4\ell}{(2\pi)^4} \frac{4(1-z)(2-z)m^2}{(\ell^2-\Delta)^3} \\
&= \frac{ig^2}{2}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z) \frac{-i}{2(4\pi)^2} \frac{4(1-z)(2-z)m^2}{\Delta} \\
&= \frac{ig^2}{2}\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \delta(1-x-y-z) \frac{-i}{2(4\pi)^2} \frac{4(1-z)(2-z)m^2}{-xyq^2 - z(1-z)m^2 + (1-z){m_W}^2}.
\end{aligned}
$$

よって

$$
\begin{aligned}
F_2(q^2=0) &= \frac{g^2}{(4\pi)^2}\frac{m^2}{{m_W}^2}\int_0^1 dz \int_0^{1-z} dy \frac{2-z}{1-z(m/m_W^2)} \\
&\approx \frac{g^2}{(4\pi)^2}\frac{m^2}{{m_W}^2}\int_0^1 dz \int_0^{1-z} dy\,(2-z) \\
&= \frac{g^2}{(4\pi)^2}\frac{m^2}{{m_W}^2}\frac{5}{6}.
\end{aligned}
$$

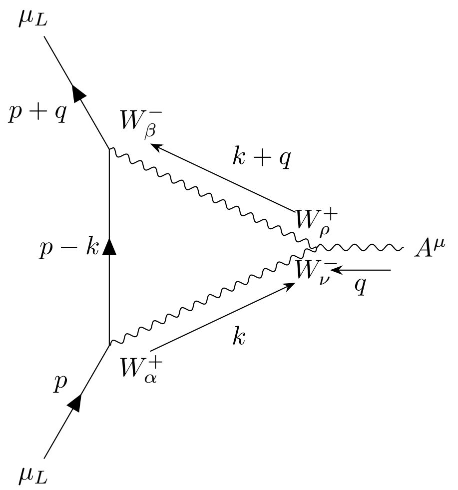
(20.91) から

$$
\frac{G_F}{\sqrt{2}} = \frac{g^2}{8{m_W}^2}
$$

なので，(6.37) から

$$
a_\mu(\nu) = F_2(q^2=0) = -\frac{G_F m^2}{8\pi^2\sqrt{2}}\frac{10}{3}.
$$

general R_ξ gauge（未計算）



頂点の 1 loop 補正は

$$
\begin{aligned}
&\bar{u}(p+q)(-ie\delta\Gamma^\mu - ie\delta\Gamma_5^\mu)u(p) \\
&= \int \frac{d^dk}{(2\pi)^d}\bar{u}(p+q)\frac{ig}{\sqrt{2}}\gamma^\beta\frac{i}{\not{p}-\not{k}}\frac{ig}{\sqrt{2}}\gamma^\alpha\frac{1-\gamma^5}{2}u(p) \\
&\quad\times\frac{-i}{k^2-m_W{}^2}\left[g_{\alpha\nu}-\frac{k_\alpha k_\nu}{k^2-\xi m_W{}^2}(1-\xi)\right]\frac{-i}{(k+q)^2-m_W{}^2}\left[g_{\beta\rho}-\frac{(k+q)_\beta(k+q)_\rho}{(k+q)^2-\xi m_W{}^2}(1-\xi)\right] \\
&\quad\times ie[g^{\nu\rho}(2k+q)^\mu + g^{\mu\nu}(-k+q)^\rho + g^{\mu\rho}(-k-2q)^\nu] \\
&= -\frac{eg^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)}\bar{u}(p+q)A_{WW}\frac{1-\gamma^5}{2}u(p) \\
&-\frac{eg^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1-\xi}{(k-p)^2(k^2-m_W{}^2)((k+q)^2-m_W{}^2)((k+q)^2-\xi m_W{}^2)} \\
&\quad\times\bar{u}(p+q)A_{12}\frac{1-\gamma^5}{2}u(p) \\
&-\frac{eg^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1-\xi}{(k-p)^2(k^2-m_W{}^2)(k^2-\xi m_W{}^2)((k+q)^2-m_W{}^2)} \\
&\quad\times\bar{u}(p+q)A_{21}\frac{1-\gamma^5}{2}u(p) \\
&-\frac{eg^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{(1-\xi)^2}{(k-p)^2(k^2-m_W{}^2)(k^2-\xi m_W{}^2)((k+q)^2-m_W{}^2)((k+q)^2-\xi m_W{}^2)} \\
&\quad\times\bar{u}(p+q)A_{22}\frac{1-\gamma^5}{2}u(p).
\end{aligned}
$$

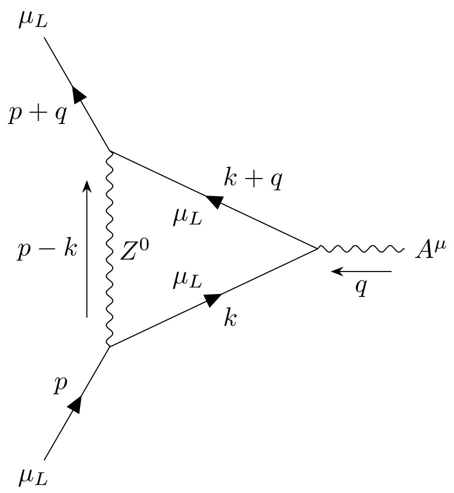
ただし

$$
\begin{aligned}
A_{12} &= -(\not{k}+\not{q})(\not{k}-\not{p})\left[(\not{k}+\not{q})(-2k-q)^\mu + \gamma^\mu(k+q)\cdot(k-q) + (\not{k}+2\not{q})(k+q)^\mu\right] \\
A_{21} &= -\not{k}(\not{k}-\not{p})\not{k}(-2k-q)^\mu + (\not{k}-\not{q})(\not{k}-\not{p})\not{k}k^\mu + \gamma^\mu(\not{k}-\not{p})\not{k}k\cdot(k+2q) \\
A_{22} &= -(\not{k}+\not{q})(\not{k}-\not{p})\not{k}\left[k\cdot(k+q)(-2k-q)^\mu + k^\mu(k+q)\cdot(k-q) + (k+q)^\mu k\cdot(k+2q)\right].
\end{aligned}
$$

ZZ diagram

$\mu\mu Z^0$ の頂点は (20.80) から

$$
\begin{aligned}
&\frac{ig}{\cos\theta_w}\gamma^\mu\left[\left(-\frac{1}{2}+\sin^2\theta_w\right)\frac{1-\gamma^5}{2}+\sin^2\theta_w\frac{1+\gamma^5}{2}\right] \\
&= \frac{ig}{4\cos\theta_w}\gamma^\mu(4\sin^2\theta_w - 1 + \gamma^5)
\end{aligned}
$$



頂点の 1 loop 補正は

$$
\begin{aligned}
&\bar{u}(p+q)(\delta\Gamma^\mu + \delta\Gamma_5^\mu)u(p) \\
&= \int \frac{d^dk}{(2\pi)^d}\bar{u}(p+q)\frac{ig}{4\cos\theta_w}\gamma^\nu(4\sin^2\theta_w - 1 + \gamma^5)\frac{i}{\not{k}+\not{q}-m}\gamma^\mu\frac{i}{\not{k}-m} \\
&\quad\times \frac{ig}{4\cos\theta_w}\gamma_\nu(4\sin^2\theta_w - 1 + \gamma^5)u(p)\frac{-i}{(p-k)^2 - m_Z{}^2} \\
&= -\frac{i}{16}\left(\frac{g}{\cos\theta_w}\right)^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)((k+q)^2-m^2)((p-k)^2-m_Z{}^2)} \\
&\quad\times \bar{u}(p+q)\gamma^\nu(4\sin^2\theta_w - 1 + \gamma^5)(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma_\nu(4\sin^2\theta_w - 1 + \gamma^5)u(p).
\end{aligned}
$$

γ^5 を含まない項は

$$
\begin{aligned}
\delta\Gamma^\mu = &-\frac{i}{16}\left(\frac{g}{\cos\theta_w}\right)^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)((k+q)^2-m^2)((p-k)^2-m_Z{}^2)} \\
&\times\left[(4\sin^2\theta_w-1)^2\gamma^\nu(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma_\nu + \gamma^\nu(\not{k}+\not{q}-m)\gamma^\mu(\not{k}-m)\gamma_\nu\right].
\end{aligned}
$$

分母は

$$
-\frac{i}{8}\left(\frac{g}{\cos\theta_w}\right)^2\int_0^1 dx\int_0^1 dy\int_0^1 dz\,\delta(1-x-y-z)\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}.
$$

ただし

$$
\begin{aligned}
k &= \ell + (zp - yq), \\
\Delta &= -z(1-z)p^2 - y(1-y)q^2 + (x+y)m^2 + zm_Z{}^2 - 2yzp\cdot q \\
&= -z(1-z)m^2 - y(1-y)q^2 + (1-z)m^2 + zm_Z{}^2 + yzq^2 \\
&= -xyq^2 + (1-z)^2m^2 + zm_Z{}^2.
\end{aligned}
$$

分子第 1 項の定数項は

$$
\begin{aligned}
&= \gamma^\nu[z\not{p} + (1-y)\not{q} + m]\gamma^\mu[z\not{p} - y\not{q} + m]\gamma_\nu \\
&= \gamma^\nu[z\not{p} + (1-y)\not{q}]\gamma^\mu[z\not{p} - y\not{q}]\gamma_\nu \\
&\quad + m\gamma^\nu[z\not{p} + (1-y)\not{q}]\gamma^\mu\gamma_\nu + m\gamma^\nu\gamma^\mu[z\not{p} - y\not{q}]\gamma_\nu + m^2\gamma^\nu\gamma^\mu\gamma_\nu
\end{aligned}
$$

$$
\begin{aligned}
&= -2[z\not{p} - y\not{q}]\gamma^\mu[z\not{p} + (1-y)\not{q}] \\
&\quad + 4m[zp + (1-y)q]^\mu + 4m[zp - yq]^\mu - 2m^2\gamma^\mu \\
&\sim -2[(y+z)\not{p} - ym]\gamma^\mu[zm + (1-y)\not{q}] + 4m[2zp^\mu + (1-2y)q^\mu] - 2m^2\gamma^\mu \\
&= -2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2yzm^2\gamma^\mu + 2y(1-y)m\gamma^\mu\not{q} \\
&\quad + 8zmp^\mu + 4(1-2y)mq^\mu - 2m^2\gamma^\mu \\
&= 2(yz-1)m^2\gamma^\mu + 8zmp^\mu + 4(1-2y)mq^\mu \\
&\quad - 2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2y(1-y)m\gamma^\mu\not{q}.
\end{aligned}
$$

ここで

$$
\begin{aligned}
\not{p}\gamma^\mu &= 2p^\mu - \gamma^\mu\not{p} \sim 2p^\mu - m\gamma^\mu, \\
\not{p}\gamma^\mu\not{q} &\sim (m - \not{q})\gamma^\mu\not{q} = (m-\not{q})(2q^\mu - \not{q}\gamma^\mu) = -2q^\mu\not{q} + 2mq^\mu + \not{q}\not{q}\gamma^\mu - m\not{q}\gamma^\mu \\
&\sim 2mq^\mu + q^2\gamma^\mu - m(2m\gamma^\mu - 2p^\mu) \\
&= (q^2 - 2m^2)\gamma^\mu + 2mp'^\mu, \\
\gamma^\mu\not{q} &= 2q^\mu - \not{q}\gamma^\mu \sim 2q^\mu - (m - \not{p})\gamma^\mu \\
&= 2q^\mu - m\gamma^\mu + \not{p}\gamma^\mu = 2q^\mu - m\gamma^\mu + 2p^\mu - \gamma^\mu\not{p} = 2q^\mu - m\gamma^\mu + 2p^\mu - m\gamma^\mu \\
&= 2p'^\mu - 2m\gamma^\mu
\end{aligned}
$$

を代入して

$$
\begin{aligned}
&\sim 2(yz-1)m^2\gamma^\mu + 8zmp^\mu + 4(1-2y)mq^\mu \\
&\quad - 2(y+z)zm(2p^\mu - m\gamma^\mu) - 2(1-y)(y+z)[(q^2-2m^2)\gamma^\mu + 2mp'^\mu] + 2y(1-y)m(2p'^\mu - 2m\gamma^\mu) \\
&= [\cdots]\gamma^\mu + 4(2-y-z)zmp^\mu - 4(1-y)zmp'^\mu + 4(1-2y)mq^\mu \\
&= [\cdots]\gamma^\mu + 2(1-z)zm(p^\mu + p'^\mu) + 2(2 - 4y - 3z + 2yz + z^2)mq^\mu \\
&= [\cdots]\gamma^\mu + 2(1-z)zm(p^\mu + p'^\mu) + 2(x-y)(x+y+1)mq^\mu \\
&\to [\cdots]\gamma^\mu + 2(1-z)zm(p^\mu + p'^\mu).
\end{aligned}
$$

分子第２項の定数項は

$$
\begin{aligned}
&= \gamma^\nu[z\not{p} + (1-y)\not{q} - m]\gamma^\mu[z\not{p} - y\not{q} - m]\gamma_\nu \\
&= \gamma^\nu[z\not{p} + (1-y)\not{q}]\gamma^\mu[z\not{p} - y\not{q}]\gamma_\nu \\
&\quad - m\gamma^\nu[z\not{p} + (1-y)\not{q}]\gamma^\mu\gamma_\nu - m\gamma^\nu\gamma^\mu[z\not{p} - y\not{q}]\gamma_\nu + m^2\gamma^\nu\gamma^\mu\gamma_\nu \\
&= -2[z\not{p} - y\not{q}]\gamma^\mu[z\not{p} + (1-y)\not{q}] \\
&\quad - 4m[zp + (1-y)q]^\mu - 4m[zp - yq]^\mu - 2m^2\gamma^\mu \\
&\sim -2[(y+z)\not{p} - ym]\gamma^\mu[zm + (1-y)\not{q}] - 4m[2zp^\mu + (1-2y)q^\mu] - 2m^2\gamma^\mu \\
&= -2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2yzm^2\gamma^\mu + 2y(1-y)m\gamma^\mu\not{q} \\
&\quad - 8zmp^\mu - 4(1-2y)mq^\mu - 2m^2\gamma^\mu \\
&= 2(yz-1)m^2\gamma^\mu - 8zmp^\mu - 4(1-2y)mq^\mu \\
&\quad - 2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2y(1-y)m\gamma^\mu\not{q} \\
&\sim 2(yz-1)m^2\gamma^\mu - 8zmp^\mu - 4(1-2y)mq^\mu \\
&\quad - 2(y+z)zm(2p^\mu - m\gamma^\mu) - 2(1-y)(y+z)[(q^2-2m^2)\gamma^\mu + 2mp'^\mu] + 2y(1-y)m(2p'^\mu - 2m\gamma^\mu) \\
&= [\cdots]\gamma^\mu - 4(2+y+z)zmp^\mu - 4(1-y)zmp'^\mu - 4(1-2y)mq^\mu
\end{aligned}
$$

$$
\begin{aligned}
&= [\cdots]\gamma^\mu - 4(3+z)zmp^\mu - 4(1-2y+z-yz)mq^\mu \\
&= [\cdots]\gamma^\mu - 2(3+z)zm(p^\mu + p'^\mu) - 2(2-4y-z-2yz-z^2)mq^\mu \\
&= [\cdots]\gamma^\mu - 2(3+z)zm(p^\mu + p'^\mu) + 2(x-y)(x+y-3)mq^\mu \\
&\to [\cdots]\gamma^\mu - 2(3+z)zm(p^\mu + p'^\mu).
\end{aligned}
$$

以上から分子の定数項は

$$
\begin{aligned}
A^0_{ZZ} &= [\cdots]\gamma^\mu + (4\sin^2\theta_w - 1)^2 2(1-z)zm(p^\mu + p'^\mu) - 2(3+z)zm(p^\mu + p'^\mu) \\
&= [\cdots]\gamma^\mu + 2\left[(4\sin^2\theta_w - 1)^2(1-z)z - (3+z)z\right] m(p^\mu + p'^\mu) \\
&= [\cdots]\gamma^\mu - 4\left[(4\sin^2\theta_w - 1)^2(1-z)z - (3+z)z\right] m^2 \frac{i\sigma^{\mu\nu}q_\nu}{2m}.
\end{aligned}
$$

よって

$$
\begin{aligned}
F_2(q^2) &= \frac{i}{16}\left(\frac{g}{\cos\theta_w}\right)^2 \int_0^1 dz \int_0^{1-z} dy \int \frac{d^d\ell}{(2\pi)^d} \frac{4\left[(4\sin^2\theta_w - 1)^2(1-z)z - (3+z)z\right] m^2}{(\ell^2 - \Delta)^3} \\
&= \frac{i}{8}\left(\frac{g}{\cos\theta_w}\right)^2 \int_0^1 dz \int_0^{1-z} dy \frac{-i}{2(4\pi)^2} \frac{4\left[(4\sin^2\theta_w - 1)^2(1-z)z - (3+z)z\right] m^2}{\Delta} \\
&= \frac{i}{8}\left(\frac{g}{\cos\theta_w}\right)^2 \int_0^1 dz \int_0^{1-z} dy \frac{-i}{2(4\pi)^2} \frac{4\left[(4\sin^2\theta_w - 1)^2(1-z)z - (3+z)z\right] m^2}{-xyq^2 + (1-z)^2m^2 + zm_Z{}^2}.
\end{aligned}
$$

$m_Z \gg m$ なので

$$
\begin{aligned}
a_\mu(Z) &= F_2(0) \\
&\approx \frac{i}{8}\left(\frac{g}{\cos\theta_w}\right)^2 \int_0^1 dz \int_0^{1-z} dy \frac{-i}{2(4\pi)^2} \frac{4\left[(4\sin^2\theta_w - 1)^2(1-z)z - (3+z)z\right] m^2}{zm_Z{}^2} \\
&\approx \frac{1}{4(4\pi)^2}\frac{m^2}{m_Z{}^2}\left(\frac{g}{\cos\theta_w}\right)^2 \int_0^1 dz \int_0^{1-z} dy \left[(4\sin^2\theta_w - 1)^2(1-z) - (3+z)\right] \\
&= \frac{1}{4(4\pi)^2}\frac{m^2}{m_Z{}^2}\left(\frac{g}{\cos\theta_w}\right)^2 \left[\frac{1}{3}(4\sin^2\theta_w - 1)^2 - \frac{5}{3}\right].
\end{aligned}
$$

(20.73)(20.91) から

$$
= \frac{G_F m^2}{8\pi^2\sqrt{2}}\left[\frac{16}{3}\sin^4\theta_w - \frac{8}{3}\sin^2\theta_w - \frac{4}{3}\right].
$$

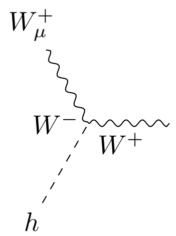
Problem 21.4: Dependence of radiative corrections on the Higgs boson mass

Feynman rules of Higgs bosons

[21.2.3] から

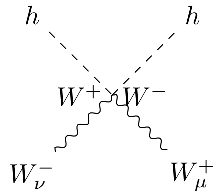
$$
i\mathcal{L}_{\text{Higgs}} = i\frac{g^2}{4}W_\mu^+ W^-\nu(v+h+i\phi^3)(v+h-i\phi^3)g^{\mu\nu} + \cdots.
$$

W^+W^-h の頂点は



$$
= i\frac{g^2 v}{2}g_{\mu\nu} = igm_W g_{\mu\nu}. \qquad [21.3.10]
$$

W^+W^-hh の頂点は h の縮約が２通りあることに注意して，

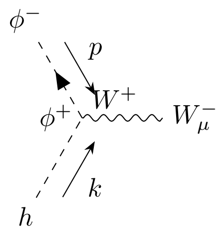


$$= 2 \cdot i\frac{g^2}{4}g_{\mu\nu} = i\frac{g^2}{2}g_{\mu\nu}.$$

[21.2.3] から

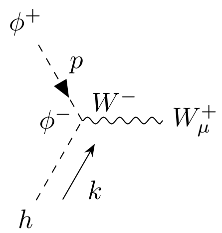
$$i\mathcal{L}_{\text{Higgs}} = i\frac{g}{2}\left[(\partial_\mu\phi^+)W^+_\nu h - (\partial_\mu h)W^+_\nu\phi^+ + (\partial_\mu\phi^-)W^-_\nu h - (\partial_\mu h)W^-_\nu\phi^-\right]g^{\mu\nu} + \cdots .$$

$W^-\phi^-h$ の頂点は



$$= i\frac{g}{2}(-ip_\mu + ik_\mu) = \frac{g}{2}(p-k)_\mu. \qquad [21.3.11]$$

$W^+\phi^+h$ の頂点は

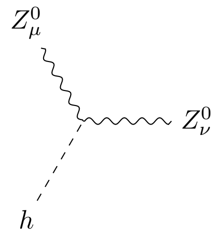


$$= i\frac{g}{2}(-ip_\mu + ik_\mu) = \frac{g}{2}(p-k)_\mu. \qquad [21.3.12]$$

[21.2.3] から

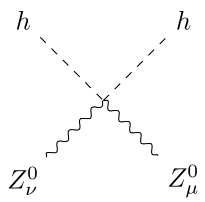
$$i\mathcal{L}_{\text{Higgs}} = i\frac{1}{8}\frac{g^2}{\cos^2\theta_w}Z^0_\mu Z^0_\nu(v+h+i\phi^3)(v+h-i\phi^3)g^{\mu\nu} + \cdots .$$

Z^0Z^0h の頂点は Z^0 の縮約が２通りあることに注意して，



$$= i\frac{g^2v}{2\cos^2\theta_w}g_{\mu\nu} = i\frac{gm_Z}{\cos\theta_w}g_{\mu\nu}.$$

Z^0Z^0hh の頂点は Z^0 の縮約が２通り，h の縮約が２通りあることに注意して，

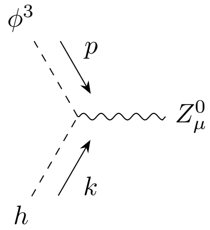


$$= i\frac{g^2}{2\cos^2\theta_w}g_{\mu\nu}.$$

[21.2.3] から

$$
\begin{aligned}
i\mathcal{L}_{\text{Higgs}} &= \frac{g}{4\cos\theta_w}\left[\partial_\mu(h+i\phi^3)(v+h-i\phi^3)-\partial_\mu(h-i\phi^3)(v+h+i\phi^3)\right]Z^0_\nu g^{\mu\nu}+\cdots \\
&= \frac{ig}{2\cos\theta_w}\left[\partial_\mu\phi^3 h-\phi^3\partial_\mu h\right]Z^0_\nu g^{\mu\nu}+\cdots
\end{aligned}
$$

$Z^0\phi^3 h$ の頂点は



$$= \frac{g}{2\cos\theta_w}(p-k)_\mu.$$

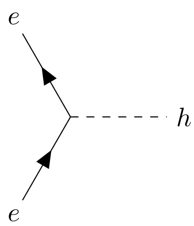
Final Project III: Dicays of the Higgs Boson

(a)

[21.3.7] から

$$\Delta\mathcal{L}_e = i\frac{g}{\sqrt{2}}\frac{m_e}{m_W}\bar{\nu}_L e_R \phi^- - \frac{g}{2}\frac{m_e}{m_W}\bar{e}_L e_R(v + h + i\phi^3) + (\text{h.c.})$$

$h\bar{e}e$ の頂点は



$$= -i\frac{g}{2}\frac{m_e}{m_W}\left(\frac{1+\gamma^5}{2} + \frac{1-\gamma^5}{2}\right) = -i\frac{g}{2}\frac{m_e}{m_W}.$$

(20.101) から quark と $SU(2)$ scalar の相互作用は

$$\Delta\mathcal{L}_q = -\lambda_d \bar{Q}_L \phi d_R - \lambda_u \epsilon^{ab}\bar{Q}_{La}\phi_b^\dagger u_R + (\text{h.c.})$$

$\Delta\mathcal{L}_q$ の第 1 項は (21.38)(21.79)(20.75) を代入して

$$\begin{aligned} -\lambda_d \bar{Q}_L \phi d_R &= -\lambda_d \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} -i\phi^- \\ (v+h+i\phi^3)/\sqrt{2} \end{pmatrix} d_R \\ &= i\lambda_d \bar{u}_L d_R \phi^- - \frac{\lambda_d}{\sqrt{2}}\bar{d}_L d_R (v + h + i\phi^3). \end{aligned}$$

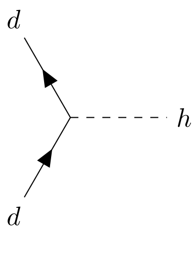
(20.63)(20.103) から

$$\lambda_d = \frac{g}{\sqrt{2}}\frac{m_d}{m_W}.$$

よって

$$-\lambda_d \bar{Q}_L \phi d_R = i\frac{g}{\sqrt{2}}\frac{m_d}{m_W}\bar{u}_L d_R \phi^- - \frac{g}{2}\frac{m_d}{m_W}\bar{d}_L d_R (v + h + i\phi^3) + (\text{h.c.})$$

$h\bar{d}d$ の頂点は

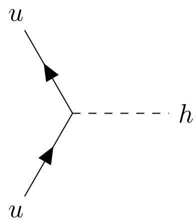


$$= -i\frac{g}{2}\frac{m_d}{m_W}.$$

$\Delta\mathcal{L}_q$ の第２項は

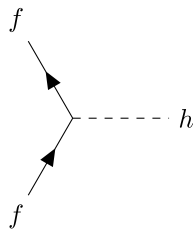
$$
\begin{aligned}
-\lambda_u \epsilon^{ab} \bar{Q}_{La} \phi_b^\dagger u_R &= -\lambda_u \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} i\phi^+ \\ (v+h-i\phi^3)/\sqrt{2} \end{pmatrix} u_R \\
&= i\lambda_u \bar{d}_L u_R \phi^+ - \frac{\lambda_u}{\sqrt{2}} \bar{u}_L u_R (v+h-i\phi^3) \\
&= i\frac{g}{\sqrt{2}} \frac{m_u}{m_W} \bar{d}_L u_R \phi^+ - \frac{g}{2} \frac{m_u}{m_W} \frac{\lambda_u}{\sqrt{2}} \bar{u}_L u_R (v+h-i\phi^3)
\end{aligned}
$$

$h\bar{u}u$ の頂点は



$$= -i\frac{g}{2}\frac{m_u}{m_W}.$$

以上から，lepton, quark いずれの場合も



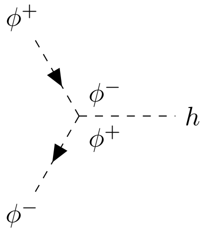
$$= -i\frac{g}{2}\frac{m_f}{m_W}.$$

(b)

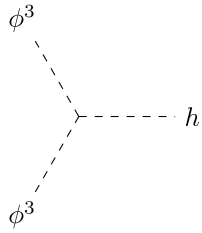
[21.2.3] と (20.63) から

$$
\begin{aligned}
\mathcal{L}_{\text{Higgs}} &= -\frac{g^2}{8}\frac{{m_h}^2}{{m_W}^2}\left[\phi^+\phi^- + \frac{(v+h)^2+(\phi^3)^2}{2}\right]^2 + \cdots \\
&= -\frac{g^2}{8}\frac{{m_h}^2}{{m_W}^2}\left[2v\phi^+\phi^- h + 2v(\phi^3)^2 h\right] + \cdots \\
&= -\frac{g^2}{2}\frac{{m_h}^2}{m_W}\phi^+\phi^- h - \frac{g^2}{2}\frac{{m_h}^2}{m_W}(\phi^3)^2 h + \cdots .
\end{aligned}
$$

$h\phi\phi$ の頂点は

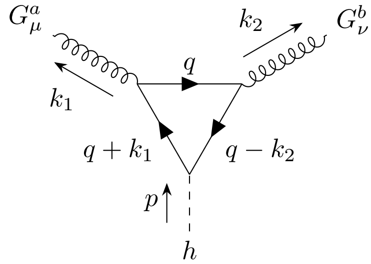


$$= -i\frac{g}{2}\frac{{m_h}^2}{m_W},$$



$$= -i\frac{g}{2}\frac{{m_h}^2}{m_W}. \qquad [\text{III.13}]$$

(c)



運動量について

$$p^\mu = (m_h, 0, 0, 0), \quad k_1^\mu = \frac{m_h}{2}(1, \sin\theta, 0, \cos\theta), \quad k_2^\mu = \frac{m_h}{2}(1, -\sin\theta, 0, -\cos\theta),$$
$${k_1}^2 = {k_2}^2 = 0, \quad k_1 \cdot k_2 = \frac{{m_h}^2}{2}, \quad \epsilon_1 \cdot k_1 = \epsilon_1 \cdot k_1 = 0.$$

不変振幅は

$$\begin{aligned} i\mathcal{M}_1 &= -i\frac{g}{2}\frac{m_q}{m_W}\int\frac{d^dq}{(2\pi)^d}(-1)\,\mathrm{Tr}\left[ig_s\not{\epsilon}_1^* t^a \frac{i}{\not{q}-m_q} ig_s \not{\epsilon}_2^* t^b \frac{i}{\not{q}-\not{k}_2-m_q}\frac{i}{\not{q}+\not{k}_1-m_q}\right] \\ &= -\frac{g{g_s}^2}{2}\frac{m_q}{m_W}\,\mathrm{Tr}(t^a t^b)\int\frac{d^dq}{(2\pi)^d}\,\mathrm{Tr}\left[\not{\epsilon}_1^*\frac{1}{\not{q}-m_q}\not{\epsilon}_2^*\frac{1}{\not{q}-\not{k}_2-m_q}\frac{1}{\not{q}+\not{k}_1-m_q}\right]. \end{aligned}$$

分母は (A.39) から

$$\begin{aligned} &\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2-{m_q}^2}\frac{1}{(q+k_1)^2-{m_q}^2}\frac{1}{(q-k_2)^2-{m_q}^2} \\ &= \int_0^1 dx\int_0^{1-x}dy\frac{2}{(\ell^2-\Delta)^3}. \end{aligned}$$

ただし

$$\begin{aligned} q &= \ell - (xk_1 - yk_2), \\ \Delta &= -x(1-x){k_1}^2 - y(1-y){k_2}^2 - 2xyk_1\cdot k_2 + {m_q}^2 = {m_q}^2 - xy{m_h}^2. \end{aligned}$$

分子

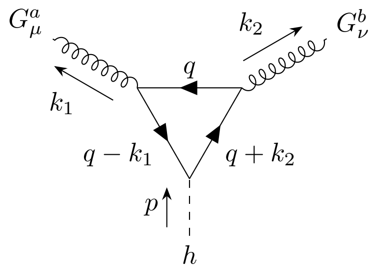
$$N = \mathrm{Tr}\left[\not{\epsilon}_1^*(\not{q}+m_q)\not{\epsilon}_2^*(\not{q}-\not{k}_2+m_q)(\not{q}+\not{k}_1+m_q)\right]$$

のうち ℓ^2 となる項は (A.27)(A.41) から

$$\begin{aligned} N_2 &= m_q\,\mathrm{Tr}\left[\not{\epsilon}_1^*\not{\epsilon}_2^*\not{\ell}\not{\ell}\right] + m_q\,\mathrm{Tr}\left[\not{\epsilon}_1^*\not{\ell}\not{\epsilon}_2^*\not{\ell}\right] + m_q\,\mathrm{Tr}\left[\not{\epsilon}_1^*\not{\ell}\not{\epsilon}_2^*\not{\ell}\right] \\ &= m_q\ell^2\,\mathrm{Tr}\left[\not{\epsilon}_1^*\not{\epsilon}_2^*\right] + 2m_q 4\epsilon_{1\mu}^*\epsilon_{2\nu}^*\left[\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 + \ell^\mu\ell^\nu\right] \\ &= 4m_q\epsilon_{1\mu}^*\epsilon_{2\nu}^*\ell^2 g^{\mu\nu} + 8m_q\epsilon_{1\mu}^*\epsilon_{2\nu}^*\left[2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2\right] \\ &= 4m_q\epsilon_{1\mu}^*\epsilon_{2\nu}^*\left[4\ell^\mu\ell^\nu - \ell^2 g^{\mu\nu}\right] \\ &\to 4m_q\epsilon_{1\mu}^*\epsilon_{2\nu}^*\left[\frac{4}{d}-1\right]\ell^2 g^{\mu\nu} \\ &= 4(\epsilon_1^*\cdot\epsilon_2^*)m_q\left[\frac{4}{d}-1\right]\ell^2. \end{aligned}$$

定数項は[*3]

$$N_0 = 2(\epsilon_1^* \cdot \epsilon_2^*) m_q (2xy{m_h}^2 - {m_h}^2 + 2{m_q}^2).$$



不変振幅は

$$\begin{aligned}
i\mathcal{M}_2 &= -i\frac{g}{2}\frac{m_q}{m_W}\int\frac{d^dq}{(2\pi)^d}(-1)\,\mathrm{Tr}\left[ig_s\not{\epsilon}_1^* t^a \frac{i}{\not{q} - m_q} ig_s \not{\epsilon}_2^* t^b \frac{i}{\not{q} + \not{k}_2 - m_q}\frac{i}{\not{q} - \not{k}_1 - m_q}\right] \\
&= -\frac{g{g_s}^2}{2}\frac{m_q}{m_W}\,\mathrm{Tr}(t^a t^b)\int\frac{d^dq}{(2\pi)^d}\,\mathrm{Tr}\left[\not{\epsilon}_1^*\frac{1}{\not{q} - m_q}\not{\epsilon}_2^*\frac{1}{\not{q} + \not{k}_2 - m_q}\frac{1}{\not{q} - \not{k}_1 - m_q}\right].
\end{aligned}$$

分母は (A.39) から

$$\begin{aligned}
&\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2 - {m_q}^2}\frac{1}{(q - k_1)^2 - {m_q}^2}\frac{1}{(q + k_2)^2 - {m_q}^2} \\
&= \int_0^1 dx \int_0^{1-x} dy \frac{2}{(\ell^2 - \Delta)^3}.
\end{aligned}$$

ただし

$$\begin{aligned}
q &= \ell + (xk_1 - yk_2), \\
\Delta &= -x(1-x){k_1}^2 - y(1-y){k_2}^2 - 2xyk_1 \cdot k_2 + {m_q}^2 = {m_q}^2 - xy{m_h}^2.
\end{aligned}$$

分子

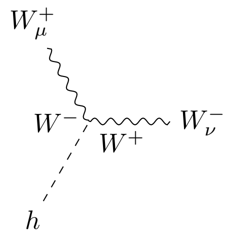
$$N = \mathrm{Tr}\left[\not{\epsilon}_1^*(\not{q} + m_q)\not{\epsilon}_2^*(\not{q} + \not{k}_2 + m_q)(\not{q} - \not{k}_1 + m_q)\right]$$

のうち ℓ^2 となる項は先程と全く同じ．定数項も計算すれば一致する．

(f)

Feynman rules of Higgs boson and photon

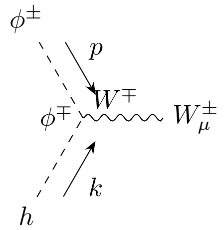
[21.3.10] から



$$= igm_W g_{\mu\nu}.$$

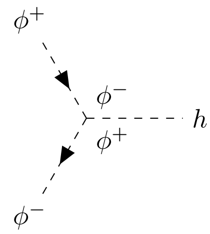
[*3] ./src/py/fp3_c.ipynb

[21.3.11][21.3.12] から



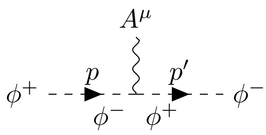
$$= \frac{g}{2}(p-k)_{\mu}.$$

[III.13] から



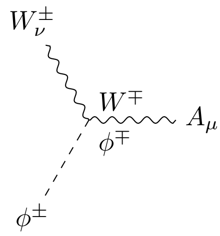
$$= -i\frac{g}{2}\frac{{m_h}^2}{m_W}.$$

[21.2.4] から



$$= ie(p+p')^{\mu}.$$

[21.3.8][21.3.9] から

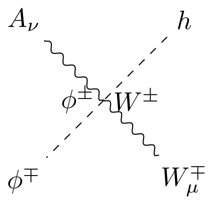


$$= \pm e m_W g^{\mu\nu}.$$

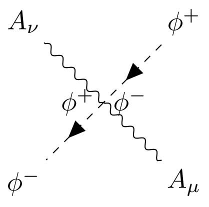
[21.2.3] から

$$\mathcal{L}_{\text{Higgs}} = i\frac{ge}{2}W^{+}_{\mu}A^{\mu}h\phi^{+} - i\frac{ge}{2}W^{-}_{\mu}A^{\mu}h\phi^{-} + e^2 A_{\mu}A^{\mu}\phi^{+}\phi^{-} + \cdots$$

なので



$$= \mp\frac{ge}{2}g_{\mu\nu},$$



$$= 2ie^2 g_{\mu\nu}.$$

$SU(2)$ の構造定数は ϵ^{abc} なので (16.6) から

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}g^2(\epsilon^{eab}A^{a}_{\kappa}A^{b}_{\lambda})(\epsilon^{ecd}A^{\kappa c}A^{\lambda d}) + \cdots$$

$(abcd) = (1133)$ となるのは

- $a=1,\, b=3,\, c=1,\, d=3$

- $a = 3,\, b = 1,\, c = 3,\, d = 1$
- $a = 1,\, b = 3,\, c = 3,\, d = 1$
- $a = 3,\, b = 1,\, c = 1,\, d = 3$

の場合．$(abcd) = (2233)$ も同様に計算すれば

$$-\frac{g^2}{2}\left(A^1_\kappa A^{\kappa 1} A^3_\lambda A^{\lambda 3} + A^2_\kappa A^{\kappa 2} A^3_\lambda A^{\lambda 3} - A^1_\kappa A^{\kappa 3} A^1_\lambda A^{\lambda 3} - A^2_\kappa A^{\kappa 3} A^2_\lambda A^{\lambda 3}\right).$$

(20.63)(20.64)(20.72) から

$$A^1_\mu = \frac{W^+_\mu + W^-_\mu}{\sqrt{2}}, \quad A^2_\mu = i\frac{W^+_\mu - W^-_\mu}{\sqrt{2}}, \qquad A^3_\mu = \frac{1}{\sin\theta_w} A_\mu$$

なので，

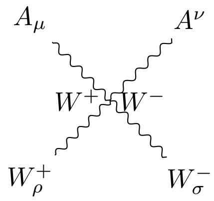
$$= g^2\left(A^\kappa W^+_\kappa A^\lambda W^-_\lambda - W^+_\kappa W^{-\kappa} A_\lambda A^\lambda\right).$$

$A^\mu A^\nu W^+_\rho W^-_\sigma$ との縮約を考える．第 1 項は

- $\kappa = \mu,\, \lambda = \nu,\, \lambda = \rho,\, \kappa = \sigma$
- $\kappa = \nu,\, \lambda = \mu,\, \lambda = \rho,\, \kappa = \sigma$

の縮約が可能で，それぞれ $\delta^\mu_\sigma \delta^\nu_\rho$ および $\delta^\mu_\rho \delta^\nu_\sigma$ となる．第 2 項は A^λ の縮約が 2 通りあるので $2g^{\mu\nu} g_{\rho\sigma}$．

以上から



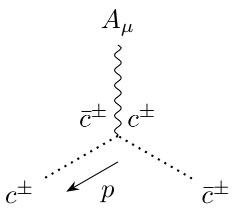
$$= ie^2\left(\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - 2g^{\mu\nu} g_{\rho\sigma}\right).$$

Feynman rules of ghosts

[21.2.2] の計算[*4]を続行する．

$$\begin{aligned}\mathcal{L}_{\text{ghost}} &= -ie\bar{c}^+ \partial^\mu A_\mu c^+ + ie\bar{c}^- \partial^\mu A_\mu c^- + \cdots \\ &\to ie(\partial^\mu \bar{c}^+) A_\mu c^+ - ie(\partial^\mu \bar{c}^-) A_\mu c^- + \cdots\end{aligned}$$

から

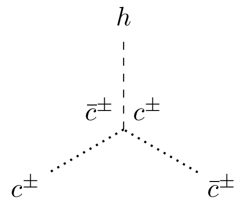


$$= \mp iep_\mu.$$

$$\mathcal{L}_{\text{ghost}} = -\frac{g}{2} m_W \bar{c}^+ c^+ h - \frac{g}{2} m_W \bar{c}^- c^- h + \cdots$$

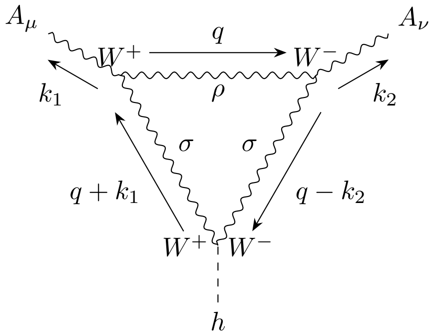
*4 `./src/py.ghost.ipynb`

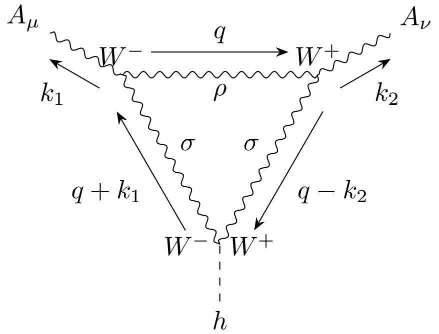
から



$$= -i\frac{g}{2}m_W.$$

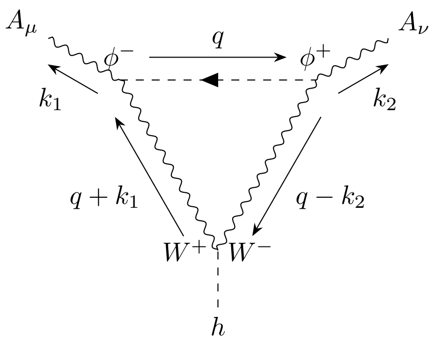
3-boson vertex diagrams

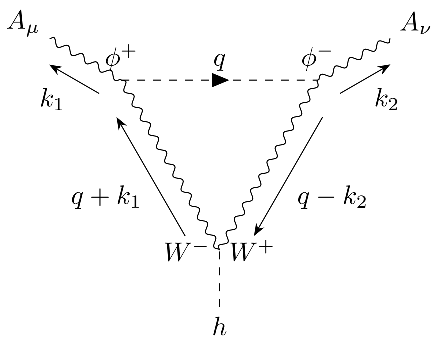




不変振幅は[*5](A.41)(A.44)(A.45) から

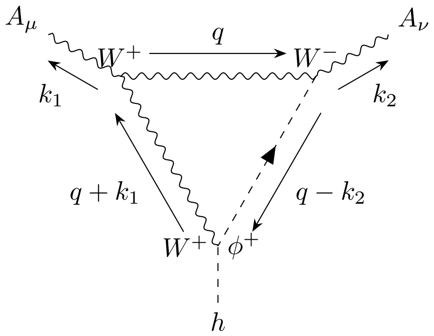
$$\begin{aligned}
&2igm_W(ie)^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)(-i)^3 2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(\mathrm{Num})}{(\ell^2-\Delta)^3}\\
&=4gm_We^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(-4xy+x+y-5)k_1\cdot k_2g^{\mu\nu}+(4d-6)\ell^\mu\ell^\nu+2\ell^2g^{\mu\nu}}{(\ell^2-\Delta)^3}\\
&=gm_We^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}4\left[(-4xy+x+y-5)\frac{{m_h}^2}{2}+\left(6-\frac{6}{d}\right)\ell^2\right]\frac{1}{(\ell^2-\Delta)^3}\\
&=\frac{i}{(4\pi)^2}gm_We^2\epsilon^*_1\cdot\epsilon^*_2\int_0^1 dx\,dy\\
&\quad\times\left[(4xy-x-y+5)\frac{{m_h}^2}{\Delta}+4\left(6-\frac{6}{d}\right)\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}\right].
\end{aligned}$$

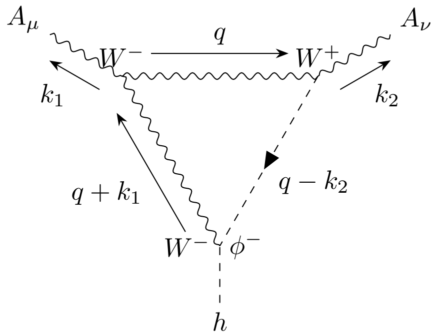




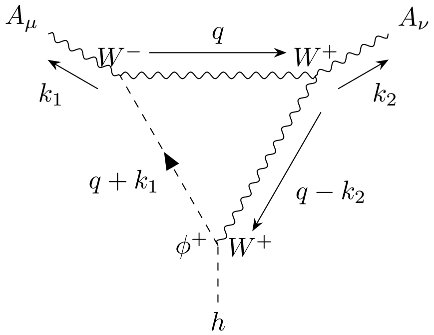
[*5] `./src/py/NonAbelian_1loop.ipynb`

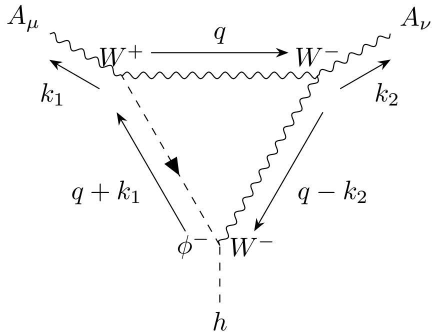
$$
\begin{aligned}
&2igm_W em_W(-em_W)g^{\mu\nu}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)(-i)^2 i2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}\\
&=-4g{m_W}^3e^2\epsilon^*_1\cdot\epsilon^*_2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}\\
&=\frac{i}{(4\pi)^2}gm_We^2\epsilon^*_1\cdot\epsilon^*_2\int_0^1 dx\,dy\,\frac{2{m_W}^2}{\Delta}.
\end{aligned}
$$





$$
\begin{aligned}
&2ie\frac{g}{2}em_W\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)(-i)^2 i2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(\mathrm{Num})}{(\ell^2-\Delta)^3}\\
&=2gm_We^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(2xy-2x+2y-2)k_1\cdot k_2g^{\mu\nu}+\ell^\mu\ell^\nu-\ell^2g^{\mu\nu}}{(\ell^2-\Delta)^3}\\
&=gm_We^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}2\left[(2xy-2x+2y-2)\frac{{m_h}^2}{2}+\left(\frac{1}{d}-1\right)\ell^2\right]\frac{1}{(\ell^2-\Delta)^3}\\
&=\frac{i}{(4\pi)^2}gm_We^2\epsilon^*_1\cdot\epsilon^*_2\int_0^1 dx\,dy\\
&\quad\times\left[-(xy-x+y-1)\frac{{m_h}^2}{\Delta}+2\left(\frac{1}{d}-1\right)\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}\right].
\end{aligned}
$$



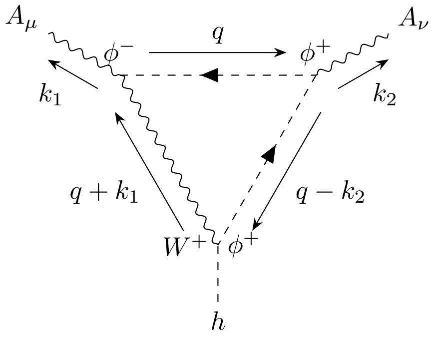


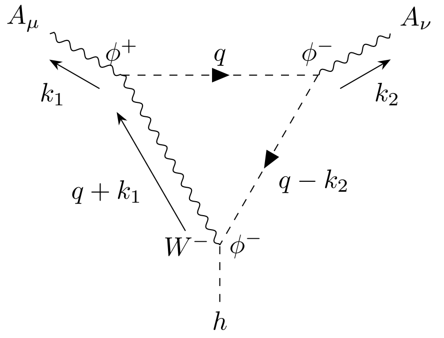
$$
\begin{aligned}
&2ie\frac{g}{2}em_W\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)(-i)^2 i2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(\mathrm{Num})}{(\ell^2-\Delta)^3}\\
&=2gm_We^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(2xy+2x-2y-2)k_1\cdot k_2g^{\mu\nu}+\ell^\mu\ell^\nu-\ell^2g^{\mu\nu}}{(\ell^2-\Delta)^3}
\end{aligned}
$$

$$= gm_We^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}2\left[(2xy+2x-2y-2)\frac{{m_h}^2}{2}+\left(\frac{1}{d}-1\right)\ell^2\right]\frac{1}{(\ell^2-\Delta)^3}$$

$$= \frac{i}{(4\pi)^2}gm_We^2\epsilon_1^*\cdot\epsilon_2^*\int_0^1 dx\,dy$$

$$\times\left[-(xy+x-y-1)\frac{{m_h}^2}{\Delta}+2\left(\frac{1}{d}-1\right)\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}\right].$$



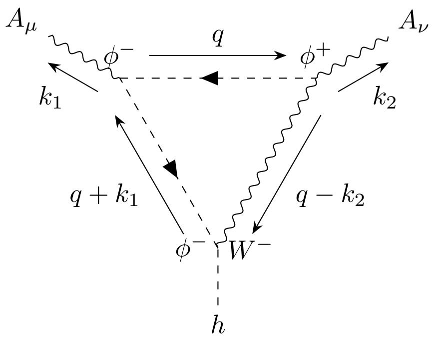


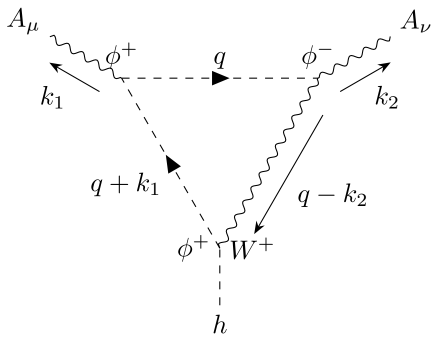
$$2ie\frac{g}{2}em_W\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)i^2(-i)2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(q-k_1-2k_2)^\mu(-2q-k_2)^\nu}{(\ell^2-\Delta)^3}$$

$$= 4gm_We^2\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^3}$$

$$= gm_We^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{4}{d}\frac{\ell^2}{(\ell^2-\Delta)^3}$$

$$= \frac{i}{(4\pi)^2}gm_We^2\epsilon_1^*\cdot\epsilon_2^*\int_0^1 dx\,dy\,\frac{4}{d}\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}.$$



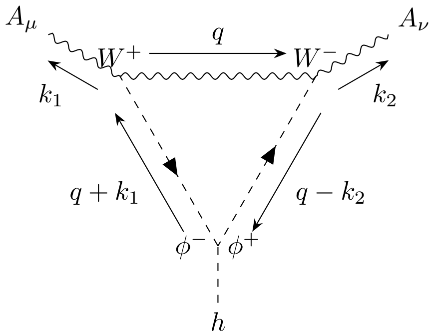


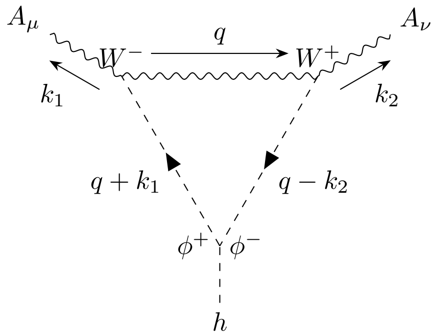
$$2ie\frac{g}{2}em_W\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)i^2(-i)2\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(-q-2k_1-k_2)^\mu(2q+k_1)^\nu}{(\ell^2-\Delta)^3}$$

$$= 4gm_We^2\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^3}$$

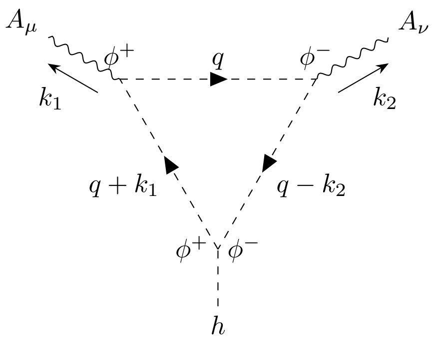
$$= gm_We^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{4}{d}\frac{\ell^2}{(\ell^2-\Delta)^3}$$

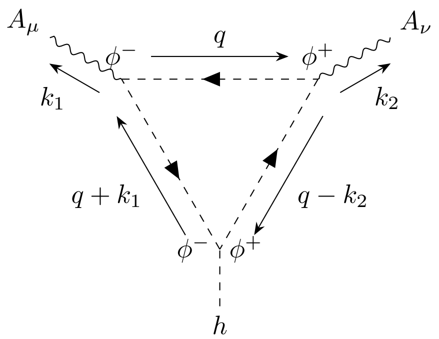
$$= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx\, dy\, \frac{4}{d}\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}.$$



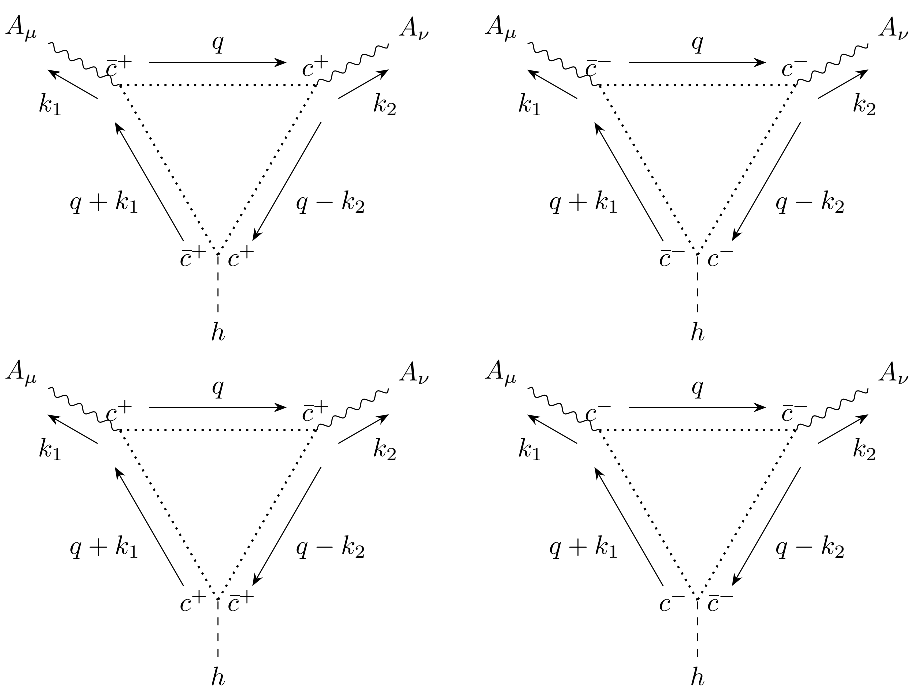


$$2\left(-i\frac{g}{2}\frac{{m_h}^2}{m_W}\right) e m_W(-e m_W)\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) g^{\mu\nu} i^2(-i)2\int_0^1 dx\, dy \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}$$
$$= -2 g m_W {m_h}^2 e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\, dy\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^3}$$
$$= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon_1^*\cdot\epsilon_2^*\int_0^1 dx\, dy\,\frac{{m_h}^2}{\Delta}$$





$$2\left(-i\frac{g}{2}\frac{{m_h}^2}{m_W}\right)(ie)^2\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2) i^3 2\int_0^1 dx\, dy\int\frac{d^d\ell}{(2\pi)^d}\frac{(2q+k_1)^\mu(2q-k_2)^\nu}{(\ell^2-\Delta)^3}$$
$$= 8g\frac{{m_h}^2}{m_W} e^2\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\int_0^1 dx\, dy\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^3}$$
$$= g m_W e^2\epsilon_\mu^*(k_1)\epsilon_\nu^*(k_2)\int_0^1 dx\, dy\int\frac{d^d\ell}{(2\pi)^d}\frac{{m_h}^2}{{m_W}^2}\frac{8}{d}\frac{\ell^2}{(\ell^2-\Delta)^3}$$
$$= \frac{i}{(4\pi)^2} g m_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\, dy\,\frac{{m_h}^2}{{m_W}^2}\frac{8}{d}\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}.$$

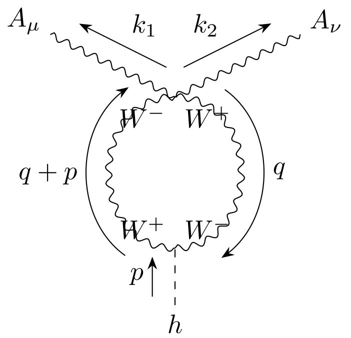


$$
\begin{aligned}
&4\left(-i\frac{g}{2}m_W\right)(ie)^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)i^3(-2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{q^\mu(q-k_2)^\nu}{(\ell^2-\Delta)^3}\\
&=-4gm_We^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{\ell^\mu\ell^\nu}{(\ell^2-\Delta)^3}\\
&=-gm_We^2\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\int_0^1 dx\,dy\int\frac{d^d\ell}{(2\pi)^d}\frac{4}{d}\frac{\ell^2}{(\ell^2-\Delta)^3}\\
&=-\frac{i}{(4\pi)^2}gm_We^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,dy\,\frac{4}{d}\frac{d}{4}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}.
\end{aligned}
$$

以上を全て足して $\Delta = m_W{}^2 - xym_h{}^2$ を代入すれば

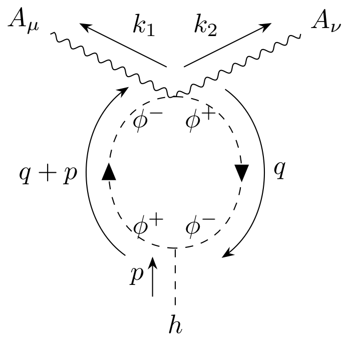
$$
\begin{aligned}
i\mathcal{M}_{333} &= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x} dy \\
&\quad\times\left[\frac{2m_W{}^2+(2xy-x-y+8)m_h{}^2}{\Delta}+\left(5d-4+\frac{2m_h{}^2}{m_W{}^2}\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}\right] \\
&= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x} dy \\
&\quad\times\left[\frac{2m_W{}^2+(2xy-x-y+8)m_h{}^2}{m_W{}^2-xym_h{}^2}-10\left(2-\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}\right. \\
&\qquad\left.+\left(16+\frac{2m_h{}^2}{m_W{}^2}\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta}\right)^{2-d/2}\right] \\
&= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x} dy \\
&\quad\times\left[\frac{2m_W{}^2+(2xy-x-y+8)m_h{}^2}{m_W{}^2-xym_h{}^2}-10\right. \\
&\qquad\left.+\left(16+\frac{2m_h{}^2}{m_W{}^2}\right)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log(m_W{}^2-xym_h{}^2)\right)\right] \\
&= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x} dy \\
&\quad\times\left[\frac{(4xy-x-y+8)m_h{}^2}{m_W{}^2-xym_h{}^2}-8+\left(16+\frac{2m_h{}^2}{m_W{}^2}\right)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log(m_W{}^2-xym_h{}^2)\right)\right].
\end{aligned}
\tag{III.14}
$$

3 + 4-boson vertex diagrams



$$
\begin{aligned}
&-(2d-2) g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta')^2} \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,(2d-2)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta'}\right)^{2-d/2} \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx \\
&\quad\times\left[-4\left(2-\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta'}\right)^{2-d/2}+6\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta'}\right)^{2-d/2}\right]
\end{aligned}
$$

$$= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx \left[-4 + 6\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta'}\right)^{2-d/2}\right].$$

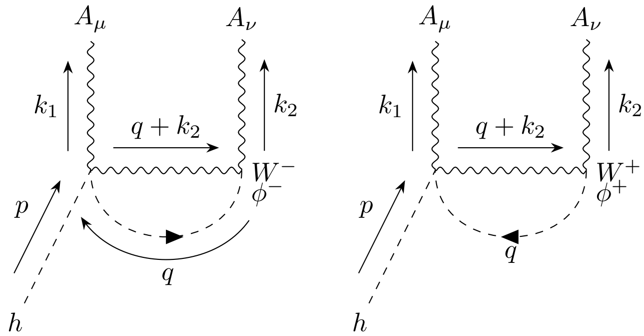


$$\begin{aligned}
&- g\frac{{m_h}^2}{m_W} e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta')^2} \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx\, \frac{{m_h}^2}{{m_W}^2}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta'}\right)^{2-d/2}.
\end{aligned}$$

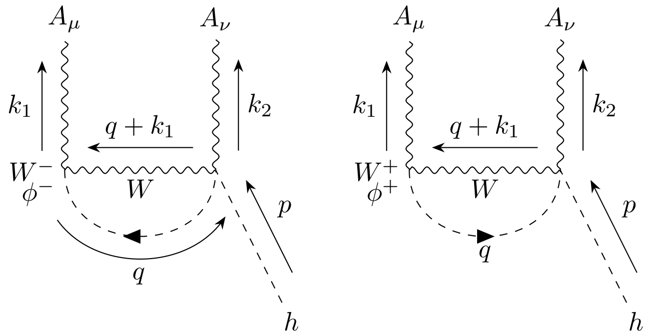
以上から $\Delta' = {m_W}^2 - x(1-x){m_h}^2$ を代入すれば

$$\begin{aligned}
i\mathcal{M}_{34} &= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx \left[-4 + \left(6+\frac{{m_h}^2}{{m_W}^2}\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta'}\right)^{2-d/2}\right] \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx \\
&\quad \times \left[-4 + \left(6+\frac{{m_h}^2}{{m_W}^2}\right)\left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log({m_W}^2 - x(1-x){m_h}^2)\right]\right].
\end{aligned} \quad \text{[III.15]}$$

$4+3$-boson vertex diagrams



$$\begin{aligned}
&- 2\frac{g}{2} m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta'')^2} \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx\, \Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta''}\right)^{2-d/2} \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx \left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log({m_W}^2)\right]
\end{aligned}$$



$$
\begin{aligned}
&-2\frac{g}{2}m_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta'')^2}\\
&=-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,\Gamma\left(2-\frac{d}{2}\right)\left(\frac{4\pi}{\Delta''}\right)^{2-d/2}\\
&=-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,\left[\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log({m_W}^2)\right]
\end{aligned}
$$

以上から

$$
i\mathcal{M}_{43}=-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)2\left[\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log({m_W}^2)\right]. \qquad \text{[III.16]}
$$

[III.14][III.15][III.16] から

$$
\begin{aligned}
i\mathcal{M}=&\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x}dy\frac{(4xy-x-y+8){m_h}^2}{{m_W}^2-xy{m_h}^2}-8\\
&+\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x}dy\left(16+\frac{2{m_h}^2}{{m_W}^2}\right)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log{m_W}^2\right)\\
&+\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x}dy\left(16+\frac{2{m_h}^2}{{m_W}^2}\right)\left(-\log(1-xy{m_h}^2/{m_W}^2)\right)\\
&-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,(-4)\\
&-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,\left(6+\frac{{m_h}^2}{{m_W}^2}\right)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log{m_W}^2\right)\\
&-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\,\left(6+\frac{{m_h}^2}{{m_W}^2}\right)\left[-\log(1-x(1-x){m_h}^2/{m_W}^2)\right]\\
&-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)2\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log{m_W}^2\right)\\
=&\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x}dy\frac{(4xy-x-y+8){m_h}^2}{{m_W}^2-xy{m_h}^2}\\
&+\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\left(8+\frac{{m_h}^2}{{m_W}^2}\right)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log{m_W}^2\right)\\
&+\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\int_0^1 dx\int_0^{1-x}dy\left(16+\frac{2{m_h}^2}{{m_W}^2}\right)\left(-\log(1-xy{m_h}^2/{m_W}^2)\right)\\
&-\frac{i}{(4\pi)^2}gm_W e^2\epsilon^*(k_1)\cdot\epsilon^*(k_2)\left(6+\frac{{m_h}^2}{{m_W}^2}\right)\left(\frac{2}{\varepsilon}-\gamma+\log 4\pi-\log{m_W}^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx\, \left(6 + \frac{{m_h}^2}{{m_W}^2}\right) \left[-\log(1 - x(1-x){m_h}^2/{m_W}^2)\right] \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) 2 \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log {m_W}^2\right) \\
&= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx\, \int_0^{1-x} dy \frac{(4xy - x - y + 8){m_h}^2}{{m_W}^2 - xy{m_h}^2} \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx\, \int_0^{1-x} dy \left(16 + \frac{2{m_h}^2}{{m_W}^2}\right) \log\left(1 - xy\frac{{m_h}^2}{{m_W}^2}\right) \\
&+ \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1)\cdot\epsilon^*(k_2) \int_0^1 dx\, \left(6 + \frac{{m_h}^2}{{m_W}^2}\right) \log\left(1 - x(1-x)\frac{{m_h}^2}{{m_W}^2}\right).
\end{aligned}
$$

$m_h \ll m_W$ なら

$$
\begin{aligned}
&\int_0^1 dx\, \int_0^{1-x} dy\, \frac{(4xy - x - y + 8){m_h}^2}{{m_W}^2 - xy{m_h}^2} \\
&\quad - \left(16 + \frac{2{m_h}^2}{{m_W}^2}\right) \int_0^1 dx\, \int_0^{1-x} dy\, \log\left(1 - xy\frac{{m_h}^2}{{m_W}^2}\right) \\
&\quad + \left(6 + \frac{{m_h}^2}{{m_W}^2}\right) \int_0^1 dx\, \log\left(1 - x(1-x)\frac{{m_h}^2}{{m_W}^2}\right) \\
&\approx \int_0^1 dx\, \int_0^{1-x} dy\, (4xy - x - y + 8)\frac{{m_h}^2}{{m_W}^2} \\
&\quad + 16 \int_0^1 dx\, \int_0^{1-x} dy\, xy\frac{{m_h}^2}{{m_W}^2} - 6\int_0^1 dx\, x(1-x)\frac{{m_h}^2}{{m_W}^2} \\
&= \frac{23}{6}\frac{{m_h}^2}{{m_W}^2} + \frac{2}{3}\frac{{m_h}^2}{{m_W}^2} - \frac{{m_h}^2}{{m_W}^2} \\
&= \frac{7}{2}\frac{{m_h}^2}{{m_W}^2}.
\end{aligned}
$$